EXACT SOLUTIONS OF INFINITE DERIVATIVE GRAVITY

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SULTAN EYLÜL ÖCAL

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submitted by **SULTAN EYLÜL ÖCAL** in partial fulfillment of the requirements for the degree of **Master of Science in Physics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** _____

Prof. Dr. Seçkin Kürkçüoğlu
Head of Department, **Physics** _____

Prof. Dr. Bayram Tekin
Supervisor, **Physics Department, METU** _____

Assoc. Prof. Dr. Ercan Kılıçarslan
Co-supervisor, **Physics Department, Usak University** _____

Examining Committee Members:

Prof. Dr. Seçkin Kürkçüoğlu
Physics Department, METU _____

Prof. Dr. Bayram Tekin
Physics Department, METU _____

Prof. Dr. Tahsin Çağrı Şişman
Astronautical Engineering, UTAA _____

Assoc. Prof. Dr. İsmet Yurduşen
Mathematics department, Hacettepe University _____

Prof. Dr. Atalay Karasu
Physics Department, METU _____

Date: 17.08.2021

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Name, Surname: Sultan Eylül Öcal

Signature :

ABSTRACT

EXACT SOLUTIONS OF INFINITE DERIVATIVE GRAVITY

Öcal, Sultan Eylül
M.S., Department of Physics
Supervisor: Prof. Dr. Bayram Tekin
Co-Supervisor: Assoc. Prof. Dr. Ercan Kılıçarslan

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Infinite Derivative Gravity (IDG) is a modified gravity theory which can avoid the singularities and Ultraviolet problem of gravity. This thesis examines the effects of IDG on these problems. First, the propagators and Newtonian potential will be examined as well as the conditions necessary for avoidance of singularities for perturbations around Minkowski background are found. Second, we study the exact pp-wave and AdS-plane wave solutions of quadratic and Infinite derivative gravity theories. We construct exact gravitational shock and impulsive wave solutions of IDG. We have demonstrated that unlike the Einstein's general relativity, even though these waves are created by linear sources having Dirac delta type singularities, singularities get smeared by the non-local interactions. All the calculations are just a review.

Keywords: Infinite Derivative Gravity, Singularities, PP-wave, AdS-plane wave, Non-local interactions

ÖZ

SONSUZ TÜREVLİ KÜTLE ÇEKİM KURAMININ TAM ÇÖZÜMLERİ

Öcal, Sultan Eylül
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Bayram Tekin
Ortak Tez Yöneticisi: Doç. Dr. Ercan Kılıçarslan

Ağustos 2021 , 54 sayfa

Sonsuz türevli kütle çekim teorisi tekillikler ve morötesi problemler içermeyen modifiye bir kütle çekim teorisidir. Tezde sonsuz türevli kütle çekim teorisinin bu problem üzerindeki etkileri incelenecektir. İlk olarak, ilerleticiler Newtonyen potansiyel incelenecek, bununla birlikte Minkowski arka planındaki pertürbasyonlar için tekilliklerden kaçınmak amacıyla gerekli koşullar irdelenecektir. İkinci kısımda, kuadratik ve sonsuz türevli kütle çekim teorilerinin tam pp-dalga ve AdS-düzlem dalga çözümleri çalışılacaktır. Sonsuz türevli kütle çekim teorisinin tam kütle çekimsel şok ve impulsif dalga çözümleri inşa edilecektir. Einstein genel görelilik teorisinin aksine bu dalgalar Dirak delta tipi tekilliklere sahip lineer kaynaklar tarafından oluşturmalarına rağmen, tekillikler lokal olmayan etkileşimler dolayısıyla ortadan kaldırılmalışdır. Tezdeki tüm hesaplamalar önceden yapılmış çalışmaların yeniden gözden geçirilmelidir.

Anahtar Kelimeler: Sonsuz türevli kütle çekimleri, Tekillikler, PP-dalgaları, AdS-uzayı, Lokal olmayan etkileşimler

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IDG	Infinite Derivative Gravity
GR	General Relativity
UV	Ultraviolet
IR	Infrared

CHAPTER 1

INTRODUCTION

Einstein's theory of General Relativity based on the Riemannian geometry which is the geometrical theory of gravitation has been very successful at describing gravity. It explains many important problems one of which is the precession of the perihelion of the Mercury which was the the first success of GR. Additionally, it examines gravitational red-shift, lensing and waves as well as predicting black holes. Therefore, one can see that GR is the most successful gravitational theory, being almost universally accepted as well as well confirmed by observations. Even though GR provides successful solutions, there is not an exact quantum completion of gravity. GR has an ultraviolet problem which is defined as cosmological and black hole types of singularities. In other words, in the classical modifications, at small scales (UV areas), the theory fails. Finite higher order theories may be helpful for the UV behavior, but result comes with a negative kinetic energy which is a ghost [1]. They are physical excitations. These excitations are represented by a negative residue in the gravitational propagator. This negative residue presents itself as negative kinetic energy which leads to instabilities at a classical level and breakdown in unitarity at the quantum level. Then, during the interactions, vacuum decays into positive and negative energy states which is known as Ostrogradsky instability. Some attempts exist to solve singularities by modifying gravity, such as the fourth derivative gravity, resulted in the ghosts, where the Hamiltonian of the theory was unbounded due to the Ostrogradsky instability. By adding higher order derivatives to the theory, instability could be avoided with the help of the appropriate choice of the some functions. Hence, infinite derivative gravity is a possible solution of resolving the ghost problem and classical singularities. It does not generate ghosts.

A particular form of IDG is free from the types of Ostragradsky instabilities, black holes and cosmological type singularities [2–8].

Due to the very complicated form of the field equations of IDG, finding exact solutions to the theory is undoubtedly a rather more difficult task. Nonetheless, remarkable progress in finding exact solutions to the theory has recently been made in finding shock and impulsive wave solutions of IDG [8, 9]. In these works, what mainly enables to attain to find the exact solutions to those highly nonlinear and nonlocal field equations is that the existing waves are described in the Kerr-Schild form, and in turn, field equations reduce to a linear and non-local differential equation which then turns into solvable forms. As a follow-up, other exact solutions of the theory have also been found in [10, 11] [1]. In this thesis, we have focused on the exact solutions of IDG and followed the articles entitled "pp-waves as exact solutions to ghost-free infinite derivative gravity" [8] and "infinite waves in ghost-free infinite derivative gravity in Anti-de-Sitter space-time" [9].

In this chapter, I am going to introduce to some background information for IDG. I am going to start with introducing some basic concepts of differential geometry and tie this to the IDG and then give the motivation for IDG, propagators and Newtonian potential. Also, to understand the other chapters, I am going to emphasize pp-waves and AdS plane waves space-times. As a subsection, I am going to talk about the curvature tensors of Kerr-Schild-Kundt class.

The second chapter will be aimed to find the exact solutions of the QG and IDG. To get the explicit solutions, I am going to choose the special form factors that satisfy ghost-freedom, which will be the same field equation that comes with the pp-wave solutions of Einstein's gravity. Also, one can see the solution of the shock wave in IDG [7].

The third chapter is related to the chapter 2. The later sections include different content which are impulsive waves in 2+1 and 3+1 dimensions and their subsections [9]. The last chapter will about the conclusions.

[1] Some exact solutions of IDG in the context of the cosmology were studied in [2, 7, 12] where a specific assumption has been made on Ricci scalar.

1.1 A Brief Information about Infinite Derivative Gravity (IDG)

1.1.1 Motivation for Infinite Derivative Gravity

The Ostrogradsky instability creates a ghost for the generic theory [2, 3, 13]. This does not apply when infinite number of derivatives exist . These were first used in string theory to avoid singularities and then applied to gravity.
The most general infinite derivative action in 4D, parity invariant, metric compatible and torsion free action is [13–15]

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R + \alpha_c[RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu} + C_{\mu\nu\rho\sigma}F_3(\Box)C^{\mu\nu\rho\sigma}]\Big], \tag{1.1}$$

with

$$F_i(\Box) = \sum_{n=0}^{\infty} f_{in}\frac{\Box^n}{M_s^{2n}}, \tag{1.2}$$

where R is the Ricci scalar, $R_{\mu\nu}$ is the Ricci tensor, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $G = \frac{1}{M_p^2}$ is the Newton's gravitational constant, $\alpha_c = \frac{1}{M_s^2}$. f's are dimensionless coefficients which play a crucial role to avoid ghost like instabilities, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator. Each $\Box$ term comes with the related M_s^2 which is a new mass scale. We work with the (-, +, +, +) metric signature. Note that the Weyl tensor vanishes precisely in a flat, or conformally flat background.
As $\alpha_c \longrightarrow 0$ or $M_s \longrightarrow \infty$, the theory reduces to Einstein's gravity with a spin 2 graviton which is massless.

1.1.2 The Newtonian Potential

One can investigate the effect of IDG on the Newtonian potential, which is a simple and important application. Even though this is a more difficult problem to solve in IDG, the result will be better (does not diverge). The Newtonian limit means static weak-field approximation. In other words, it can be described as weak fields for which the sources are static.
We consider metric fluctuations around the Minkowski space-time [15],

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1. \tag{1.3}$$

The assumption that $h_{\mu\nu}$ is small enough allows us to ignore everything except the first order. Hence, one can write an equation as,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{1.4}$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\rho}h_{\alpha\rho}$.

$$\mathcal{L} = \sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}RF_1(\Box)R + \frac{1}{2}R_{\mu\nu}F_2(\Box)R^{\mu\nu} + \frac{1}{2}C_{\mu\nu\rho\sigma}F_3(\Box)C^{\mu\nu\rho\sigma} + \mathcal{L}_{matter}\right] \tag{1.5}$$

which is the Lagrangian density, where M_p is the Planck mass, R is the scalar curvature, $R_{\mu\nu}$ is the Ricci tensor and the last one $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. One can see that a, b, c, d and f are nonlinear functions of the derivative operators which reduce in the limit to the constants values of a, b, c and d. The function $f(\Box)$ occurs only in higher or infinite derivative theories. Here, new relations are necessary,

$$a(\Box)R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}c(\Box)R^L - \frac{1}{2}f(\Box)\partial_\mu\partial_\nu R^L = \kappa T_{\mu\nu}, \tag{1.6}$$

where L is the linearization as well as non-linear functions can be defined as,

$$\begin{aligned} a(\Box) &= 1 + M_p^{-2}(F_2(\Box) + 2F_3(\Box))\Box, \\ c(\Box) &= 1 - M_p^{-2}(4F_1(\Box) + F_2(\Box) - \frac{2}{3}F_3(\Box))\Box, \\ f(\Box) &= M_p^{-2}(4F_1(\Box) + 2F_2(\Box) + \frac{4}{3}F_3(\Box)). \end{aligned} \tag{1.7}$$

Hence, the field equations could be derived easily,

$$\begin{aligned} \frac{1}{2}[a(\Box)(\Box h_{\mu\nu} - \partial_\sigma(\partial_\mu h^\sigma_\nu + \partial_\nu h^\sigma_\mu)) + c(\Box) \times (\partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial_\sigma\partial_\rho h^{\sigma\rho} - \eta_{\mu\nu}\Box h) \\ f(\Box)\partial_\mu\partial_\nu\partial_\sigma\partial_\rho h^{\sigma\rho}] = -\kappa T_{\mu\nu}. \end{aligned} \tag{1.8}$$

We are going to examine the scalar potentials in the non-local theories for short distances. Then, one can solve the linearized modified Einstein's equations for a point sources,

$$T_{\mu\nu} = \rho\delta^0_\mu\delta^0_\nu = m\delta^3(\vec{r})\delta^0_\mu\delta^0_\nu. \tag{1.9}$$

Because the Newtonian potentials are static,

$$(a - 3c)\Box h + (4c - 2a + f)\partial_\mu\partial_\nu h^{\mu\nu} = \kappa\rho, \tag{1.10}$$

$$a\Box h_{00} + c\Box h - c\partial_\mu\partial_\nu h^{\mu\nu} = -\kappa\rho, \tag{1.11}$$

For the static metric, the above equations are simplified as,

$$ds^2 = -(1-2\phi)dt^2 + (1+2\psi)dr^2, \tag{1.12}$$

where $\phi(r)$ and $\psi(r)$ are potentials.

$$2(a-3c)[\nabla^2\phi - 4\nabla^2\psi] = \kappa\rho, \tag{1.13}$$

$$2(c-a)\nabla^2\phi - 4c\nabla^2\psi = -\kappa\rho. \tag{1.14}$$

We are going to look at the functions $c(\Box)$ and $a(\Box)$, there are no ghosts and $1/r$ divergence at short distances (UV). For the situation $f = 0(a = c)$, the Newtonian potentials can be solved $\psi = \phi$. This choice ensures that the theory has not additional degrees of freedom other than massless graviton.

$$4a(\nabla^2)\nabla^2\phi = \kappa\rho = \kappa m\delta^3(\vec{r}). \tag{1.15}$$

Here $\nabla^2 = \partial_i\partial^i$ is the Laplace operator. Now, we understand that to avoid the ghost problem, $a(\Box)$ will be an exponential of an entire function. Consider the following functional dependence relation,

$$a(\Box) = e^{-\frac{\Box}{M^2}}. \tag{1.16}$$

After doing some algebra, in order to reduce the graviton propagator to the GR which is a special case $a(\Box) = c(\Box)$, one can express the potential as [3],

$$\begin{aligned}\phi(r) = \psi(r) &= -\frac{km}{(2\pi)^2 r}\int_0^\infty \frac{dp}{p}\frac{sin(pr)}{a(-p^2)},\\ &= \frac{-km}{(2\pi)^2 r}\int_0^\infty \frac{dp}{p}e^{-\frac{p^2}{m^2}}sin(pr),\\ &= \frac{Gm}{r}erf(\frac{mr}{2}),\end{aligned} \tag{1.17}$$

where $erf(r)$ is the error function given by the integral,

$$erf(r) = \frac{2}{\sqrt{\pi}}\int_0^r e^{-p^2}dp. \tag{1.18}$$

Now, let us consider the small and large distance behaviours of Newtonian potential. For the large distances as $r \to \infty$, $erf(r) \to 1$, and potential takes the following form

$$\phi(r) = -\frac{Gm}{r} \tag{1.19}$$

which reproduces the pure GR result. On the other hand, for the small distances, as $r \to 0$, $erf(r) \to r$, potential reduces to

$$\phi(r) = -\frac{GmM}{\sqrt{\pi}}. \tag{1.20}$$

Observe that Newtonian potential is constant and hence potential is regular although there is a dirac delta function type of singularity.

1.1.3 Propagators

One can realize that ghost like degrees of freedom could be still avoided if the derivative order is infinite.[16] By modifying the quadratic part of the action as $e^{\Box/M^2}$, one can prevent the presence of the useless poles. The existence of non-polynomial derivatives makes the action non-local, and this is used to deal with ultraviolet divergences in the loop integrals. This is known as non-local or infinite derivative field theories.

The propagator for IDG is [17],

$$\Pi(k^2)_{IDG} = \frac{P^{(2)}}{(k^2)a(-k^2)} + \frac{P_s^{(0)}}{k^2(a(-k^2) - 3c(-k^2))}, \tag{1.21}$$

where

$$a(\bar{\Box}) = 1 + M_p^{-2}(F_2(\bar{\Box}) + 2F_3(\bar{\Box}))\bar{\Box}, \tag{1.22}$$

$$c(\bar{\Box}) = 1 + M_p^{-2}(-4F_1(\bar{\Box}) - F_2(\bar{\Box}) + \frac{2}{3}F_3(\bar{\Box}))\bar{\Box}, \tag{1.23}$$

$$\Box \longrightarrow -k^2 = -k_\mu k^\mu. \tag{1.24}$$

I want to show that there are no ghosts, which are generated because of the negative residues in the propagator. Hence, we can set the simplest choice [18],

$$a(-k^2) = c(-k^2) = e^{\gamma(-k^2)}. \tag{1.25}$$

No extra scalar degrees of freedom. This equation ensures that the theory has not got an additional degrees of freedom other than massless spin -2 graviton.
Then, the IDG propagator simplifies as [13]

$$\Pi(k^2)_{IDG} = \frac{1}{a(-k^2)}\left(\frac{P^{(2)}}{k^2} - \frac{P_s^{(0)}}{2k^2}\right) = \frac{1}{a(-k^2)}. \tag{1.26}$$

where operators $P^{(2)}$ and $P_s^{(0)}$ are Barnes-Rivers spin projection operators. The most important point is to avoid ghost-like instabilities which means that the propagator does not have any extra degrees of freedom. $a(k^2)$ could be chosen to be an exponential function as $a(k^2) = e^{\gamma(\frac{k^2}{M^2})}$. This choice tells us that the propagator has no poles that are additional unlike massless graviton.

1.2 PP-Wave Spacetimes

One can consider the pp-wave metric described by Kerr-Schild form ($g_{\mu\nu} = \eta_{\mu\nu} + fk_\mu k_\nu$) [19–21]

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu, \tag{1.27}$$

where $\eta_{\mu\nu}$ is the flat (Minkowski) metric and covariantly constant null vector λ_μ satisfies,

$$\lambda^\mu\lambda_\mu = 0, \quad \nabla_\mu\lambda_\nu = 0, \tag{1.28}$$

$$\lambda^\mu\partial_\mu H = 0. \tag{1.29}$$

The null vector is non-expanding, non-twisting and shear-free. To find the pp wave solution of higher derivative gravity theory, I am going to calculate the Riemann, Ricci and scalar curvature after finding Christoffel connection. I am going to start with using the general equation of Christoffel connection,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{1.30}$$

Let us use (1.27) and inverse of this equation to obtain the Christoffel connection, and then substituting these into the general form of Christoffel connection, one get

$$\begin{aligned}\Gamma^\sigma_{\mu\nu} &= \frac{1}{2}(\eta^{\sigma\lambda} - 2H\lambda^\sigma\lambda^\lambda)[2\partial_\mu(H\lambda_\lambda\lambda_\nu) + 2\partial_\nu(H\lambda_\mu\lambda_\lambda) - 2\partial_\lambda(H\lambda_\mu\lambda_\nu)] \\ &= \eta^{\sigma\lambda}(\lambda_\lambda\lambda_\nu\partial_\mu H + \lambda_\mu\lambda_\lambda\partial_\nu H - \lambda_\mu\lambda_\nu\partial_\lambda H) \\ &+\eta^{\sigma\lambda}H[\partial_\mu(\lambda_\lambda\lambda_\nu) + \partial_\nu(\lambda_\mu\lambda_\nu) - \partial_\lambda(\lambda_\mu\lambda_\nu)] + \mathcal{O}(H^2),\end{aligned} \tag{1.31}$$

and one can simplify the connection equation by using the covariant derivative of these,

$$\nabla_\mu(\lambda_\lambda\lambda_\nu) = \partial_\mu(\lambda_\lambda\lambda_\nu) - \Gamma^\sigma_{\mu\lambda}\lambda_\sigma\lambda_\nu - \Gamma^\sigma_{\mu\nu}\lambda_\lambda\lambda_\sigma = 0, \tag{1.32}$$

$$\nabla_\nu(\lambda_\mu\lambda_\lambda) = \partial_\nu(\lambda_\mu\lambda_\lambda) - \Gamma^\sigma_{\nu\mu}\lambda_\sigma\lambda_\lambda - \Gamma^\sigma_{\nu\lambda}\lambda_\mu\lambda_\sigma = 0, \tag{1.33}$$

$$\nabla_\lambda(\lambda_\mu\lambda_\nu) = \partial_\lambda(\lambda_\mu\lambda_\nu) - \Gamma^\sigma_{\lambda\mu}\lambda_\sigma\lambda_\nu - \Gamma^\sigma_{\lambda\nu}\lambda_\mu\lambda_\sigma = 0. \tag{1.34}$$

By the help of these calculation, the result can be written as

$$\Gamma^\sigma_{\mu\nu} = \eta^{\sigma\lambda}(\lambda_\lambda\lambda_\nu\partial_\mu H + \lambda_\mu\lambda_\lambda\partial_\nu H - \lambda_\mu\lambda_\nu\partial_\lambda H),$$

$$= (\lambda^\sigma\lambda_\nu\partial_\mu H + \lambda^\sigma\lambda_\mu\partial_\nu H - \lambda_\mu\lambda_\nu\eta^{\sigma\beta}\partial_\beta H), \tag{1.35}$$

which satisfies $\lambda_\sigma\Gamma^\sigma_{\mu\nu} = 0$ and $\lambda^\mu\Gamma^\sigma_{\mu\nu} = 0$.
One can show that these two equations are satisfied.
The first one is,

$$\lambda_\sigma\Gamma^\sigma_{\mu\nu} = \overbrace{\lambda_\sigma\lambda^\sigma}^{nullvector}\ \lambda_\mu\partial_\nu H + \overbrace{\lambda_\sigma\lambda^\sigma}^{nullvector}\ \lambda_\nu\partial_\mu H - \lambda_\mu\lambda_\nu\overbrace{\lambda_\sigma\eta^{\sigma\beta}\partial_\beta H}^{\lambda^\beta\partial_\beta H=0} = 0. \tag{1.36}$$

Second one is,

$$\lambda^\mu\Gamma^\sigma_{\mu\nu} = \overbrace{\lambda^\mu\lambda^\sigma}^{nullvector}\ \lambda_\mu\partial_\nu H + \overbrace{\lambda^\mu\partial_\mu H}^{=0}\lambda^\sigma\lambda_\nu - \overbrace{\lambda^\mu\lambda_\mu}^{nullvector}\ \lambda_\nu\eta^{\sigma\beta}\partial_\beta H = 0. \tag{1.37}$$

After these calculations, one can find the necessary terms, the general form of the Riemann tensor as

$$R_{\rho\sigma\mu\nu} = \frac{\partial\Gamma_{\sigma\nu\rho}}{\partial x^\mu} - \frac{\partial\Gamma_{\sigma\nu\rho}}{\partial x^\nu} + \Gamma_{\rho\nu\delta}\Gamma^\delta_{\sigma\mu} - \Gamma_{\rho\mu\delta}\Gamma^\delta_{\sigma\mu}. \tag{1.38}$$

One gets the Riemann tensor by using the equations (1.27) and (1.35),

$$\begin{gathered}
R_{\rho\sigma\mu\nu} = (\eta_{\rho\lambda} + 2H\lambda_\rho\lambda_\lambda)(\partial_\mu\Gamma^\lambda_{\nu\sigma} - \partial_\nu\Gamma^\lambda_{\mu\sigma}) \\
(\eta_{\rho\lambda} + 2H\lambda_\rho\lambda_\lambda)[\partial_\mu(\lambda^\lambda\lambda_\nu\partial_\sigma H + \lambda^\lambda\lambda_\sigma\partial_\nu H - \lambda_\nu\lambda_\sigma\eta^{\lambda\beta}\partial_\beta H) - \partial_\nu(\lambda^\lambda\lambda_\mu\partial_\sigma H \\
+\lambda^\lambda\lambda_\sigma\partial_\mu H - \lambda_\mu\lambda_\sigma\eta^{\lambda\beta}\partial_\beta H)] \\
= \lambda_\rho\lambda_\nu\partial_\mu\partial_\sigma H + \lambda_\rho\lambda_\sigma\partial_\mu\partial_\nu H - \lambda_\nu\lambda_\sigma\eta^\beta_\rho\partial_\beta\partial_\mu H - \lambda_\rho\lambda_\mu\partial_\nu\partial_\sigma H - \lambda_\rho\lambda_\sigma\partial_\mu\partial_\nu H \\
+\lambda_\mu\lambda_\sigma\eta^\beta_\rho\partial_\beta\partial_\nu H.
\end{gathered} \tag{1.39}$$

Finally, the Riemann tensor can be found as,

$$R_{\rho\sigma\mu\nu} = \lambda_\rho\lambda_\nu\partial_\sigma\partial_\mu H + \lambda_\sigma\lambda_\mu\partial_\rho\partial_\nu H - \lambda_\rho\lambda_\mu\partial_\sigma\partial_\nu H - \lambda_\sigma\lambda_\nu\partial\rho\partial_\mu H. \tag{1.40}$$

The second necessary tensor is the Ricci tensor and to obtain this one I am going to introduce a new equation,

$$R_{\sigma\nu} = \eta^{\rho\mu} R_{\rho\sigma\mu\nu}. \tag{1.41}$$

Hence, the Ricci tensor is

$$R_{\mu\nu} = -\lambda_\mu\lambda_\nu\partial^2 H. \tag{1.42}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The scalar curvature is zero for the metric of the equation (1.27).

The last one is the Weyl (or Conformal) tensor. The general form of Weyl tensor can be found as

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} + \frac{(g_{\rho\nu}R_{\mu\sigma} + g_{\sigma\mu}R_{\nu\rho} - g_{\rho\mu}R_{\nu\sigma} - g_{\sigma\nu}R_{\mu\rho})}{(n-2)} + \frac{(g_{\rho\mu}g_{\mu\sigma} - g_{\rho\nu}g_{\mu\sigma})R}{(n-1)(n-2)} \\ &= \lambda_\rho\lambda_\nu\partial_\sigma\partial_\mu H + \lambda_\sigma\lambda_\mu\partial_\rho\partial_\nu H - \lambda_\rho\lambda_\mu\partial_\sigma\partial_\nu H - \lambda_\sigma\lambda_\nu\partial_\rho\partial_\mu H \\ &\quad -\frac{1}{2}(\eta_{\rho\nu}\lambda_\mu\lambda_\sigma + \eta_{\sigma\mu}\lambda_\nu\lambda_\rho - \eta_{\rho\mu}\lambda_\nu\lambda_\sigma - \eta_{\sigma\nu}\lambda_\mu\lambda_\rho)\partial^2 H \end{aligned} \tag{1.43}$$

The traceless part of the Riemann tensor is the Weyl tensor. Additionally, any contraction with λ^μ vector yields zero.

$$\lambda^\mu C_{\rho\sigma\mu\nu} = 0, \quad \lambda^\mu R_{\rho\sigma\mu\nu} = 0, \quad \lambda^\mu R_{\mu\nu} = 0 \tag{1.44}$$

Two tensors can be represented as,

$$[R^{n_0}(\nabla^{n_1}R)(\nabla^{n_2}R).....(\nabla^{n_k}R)]_{\mu\nu} \tag{1.45}$$

where $\bigtriangledown^{n_i}R_i$ represents the $(0, n_i + 4)$ rank tensor and it builds from the Riemann tensor. The general form of two tensors composed of a linear combination of $R_{\mu\nu}$ and $\Box^n R_{\mu\nu}$ which is an important property. I will give a proof for this property.

For the first part is that λ being a vector cannot make a nonzero contraction. Hence, two features are remarkable to understand,

$$\lambda^\mu\partial_\mu H = 0, \tag{1.46}$$

$$\nabla_\nu\lambda^\mu = 0. \tag{1.47}$$

This equation implies that λ is covariantly constant. Hence, λ contraction with $\nabla^2 H$ is zero.

$$\lambda^\mu \nabla_\nu \partial_\mu H = 0. \tag{1.48}$$

One can see that λ contraction with $\nabla^2 H$ and other λ's (null vector) are also zero. After this part, the contraction λ vector with $\nabla^n H$'s reduces to a lower order terms,

$$\lambda^{\mu_j} \nabla_{\mu_1} \nabla_{\mu_2} ... \nabla_{\mu_j} ... \nabla_{\mu_n} H = \nabla_{\mu_1} (\lambda^{\mu_j} \nabla_{\mu_2} ... \nabla_{\mu_j} ... \nabla_{\mu_n} H), \tag{1.49}$$

which is the first step and then,

$$\lambda^{\mu_j} \nabla_{\mu_1} \nabla_{\mu_2} ... \nabla_{\mu_j} ... \nabla_{\mu_n} H = \lambda^{\mu_j} \nabla_{\mu_2} \nabla_{\mu_1} ... \nabla_{\mu_j} ... \nabla_{\mu_n} H. \tag{1.50}$$

This is the last part that the λ cannot make a nonzero contraction.

The only nonzero part,

$$\nabla_{\mu_1} \nabla_{\mu_2} ... \nabla_{\mu_{2n-2}} \nabla^\alpha \nabla^\beta R_{\mu\alpha\nu\beta} = \nabla_{\mu_1} \nabla_{\mu_2} ... \nabla_{\mu_{2n-2}} \Box R_{\mu\nu}, \tag{1.51}$$

where we have used the Bianchi identity on the Riemann tensor for the pp wave metric defined as [19]

$$[\nabla^{2n} R]_{\mu\nu} = \Box^n R_{\mu\nu}. \tag{1.52}$$

1.3 AdS-Plane Wave Spacetimes

AdS plane-waves are a special kind of gravitational waves propagating along AdS-space[22].
D- dimensional metric in Kerr-Schild form given as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2H\lambda_\mu \lambda_\nu, \tag{1.53}$$

and the inverse metric given as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - 2H\lambda^\mu \lambda^\nu, \tag{1.54}$$

which is an exact form. Let us note that the similarity with a perturbation where the metric perturbation is defined as $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ and at the linearized level, the

inverse metric becomes $g_{\mu\nu} = \bar{g}_{\mu\nu} - h_{\mu\nu}$. Here, $\bar{g}_{\mu\nu}$ is the AdS metric. Additionally, the following relations are satisfied

$$\begin{aligned} \lambda^\mu \lambda_\mu &= 0, \\ \nabla_\mu \lambda_\nu &= \xi_{(\mu}\lambda_{\nu)}, \\ \xi_\mu \lambda^\mu &= 0, \\ \lambda^\mu \partial_\mu H &= 0. \end{aligned} \tag{1.55}$$

Hence, the Kerr-Schild metric is a member of Kundt class

$$\xi^\mu \nabla_\mu \lambda_\nu = \xi^\mu \nabla_\nu \lambda_\mu = \frac{1}{2}\lambda_\nu \xi^\mu \xi_\mu, \tag{1.56}$$

and

$$\bar{\nabla}^\nu(\xi^\mu \lambda_\mu) = 0 \overbrace{\longrightarrow}^{symmetric} \lambda^\mu \bar{\nabla}_\nu \xi_\mu \overbrace{=}^{antisymmetric} -\xi^\mu \bar{\nabla}_\nu \lambda^\mu. \tag{1.57}$$

To reach the equation (1.56), one should think about the theorem and its proof, i.e.;

$$\lambda^{\mu_j}(\prod_{i=1}^{n-1} \nabla_{\mu_i})\xi_{\mu_n} \tag{1.58}$$

is reducible. We are going to show how to prove this identity

$$\nabla_\mu \lambda_\nu = \xi_{(\mu}\lambda_{\nu)} \tag{1.59}$$

ξ satisfies the identities,

$$\lambda^{\mu_1} \nabla_{\mu_1} \xi_{\mu_2} = -\lambda_{\mu_2}(\frac{1}{4}\xi^{\mu_1}\xi_{\mu_1} - \frac{R}{12}), \tag{1.60}$$

and

$$\lambda^{\mu_2} \nabla_{\mu_1} \xi_{\mu_2} = -\frac{1}{2}\lambda_{\mu_1}\xi^{\mu_2}\xi_{\mu_2}. \tag{1.61}$$

The Christoffel connection of the metric is

$$\bar{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \Omega^\mu_{\alpha\beta}, \tag{1.62}$$

$\Omega^\mu_{\alpha\beta}$ which is the background metric of the Christoffel connection $\bar{g}_{\mu\nu}$.

$$\Omega^\mu_{\alpha\beta} = \bar{\nabla}_\alpha(H\lambda^\mu\lambda_\beta) + \bar{\nabla}_\beta(H\lambda^\mu\lambda_\alpha) - \bar{\nabla}^\mu(H\lambda_\alpha\lambda_\beta). \tag{1.63}$$

One can also easily show that $\Omega^\mu_{\alpha\beta}$ satisfies the properties,

$$\Omega^\mu_{\mu\beta} = \bar{\nabla}_\mu(H\lambda^\mu\lambda_\beta) + \bar{\nabla}_\beta(H\overbrace{\lambda^\mu\lambda_\mu}^{=0}) - \bar{\nabla}^\mu(H\lambda_\mu\lambda_\beta) = 0, \tag{1.64}$$

$$\lambda_\mu \Omega^\mu_{\alpha\beta} = \overbrace{\lambda_\mu \lambda^\mu}^{=0} \bar{\nabla}_\alpha H \lambda_\beta + \overbrace{\lambda_\mu \lambda^\mu}^{=0} \bar{\nabla}_\beta H \lambda_\alpha - \bar{\nabla}^\mu \lambda_\mu H \lambda_\alpha \lambda_\beta = 0, \tag{1.65}$$

$$\lambda^\alpha \Omega^\mu_{\alpha\beta} = \overbrace{\bar{\nabla}_\alpha \lambda^\alpha \lambda_\beta}^{=0} H \lambda^\mu + \bar{\nabla}_\beta H \lambda^\mu \overbrace{\lambda^\alpha \lambda_\alpha}^{=0} - \bar{\nabla}^\mu H \overbrace{\lambda_\alpha \lambda^\alpha}^{=0} \lambda_\beta = 0. \tag{1.66}$$

And thus, the covariant derivative of λ^μ reduces to the covariant derivative with respect to the background AdS metric,

$$\bar{\nabla}_\mu \lambda_\rho = \nabla_\mu \lambda_\rho. \tag{1.67}$$

Using the Ricci identity in the form

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \lambda^\mu = \bar{R}_{\nu\sigma} \lambda^\sigma, \tag{1.68}$$

with

$$\bar{\nabla}_\mu \lambda^\mu = 0, \tag{1.69}$$

$$(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{\nabla}_\nu \bar{\nabla}_\mu) \lambda^\mu = -\frac{3}{l^2} \bar{g}_{\nu\sigma} \lambda^\sigma = -\frac{3}{l^2} \lambda_\nu. \tag{1.70}$$

The first one of the left hand side comes from equation (2.28) and the second one comes from equation (1.69) as zero. Hence, we can reach the final part,

$$\bar{\Box} \lambda_\nu = -\frac{3}{l^2} \lambda_\nu. \tag{1.71}$$

I will show how to solve this last equation,

$$\bar{\Box} \lambda_\nu = \bar{\nabla}^\sigma \bar{\nabla}_\sigma \lambda_\nu = \bar{\nabla}^\sigma \bar{\nabla}_\nu \lambda_\sigma = [\bar{\nabla}^\sigma, \bar{\nabla}_\nu] \lambda_\sigma = \bar{R}^\sigma_{\nu\sigma}{}^\rho \lambda_\rho. \tag{1.72}$$

Also, the left hand side yields a new equation,

$$\lambda^\mu \bar{\nabla}_\mu \xi_\nu = -\lambda_\nu [\bar{\nabla}_\mu \xi^\mu + \frac{1}{2} \xi^\mu \xi_\mu + \frac{6}{l^2}]. \tag{1.73}$$

To obtain the curvature tensor, we are going to find the $\Omega^\mu{}_{\alpha\beta}$ part of the Christoffel connection which is linear in H ,

$$\Omega^\mu{}_{\alpha\beta} = -\lambda_\alpha \lambda_\beta \partial^\mu H + 2H \lambda^\mu \lambda_{(\alpha} \partial_{\beta)} + 2H \lambda^\mu \lambda_{(\alpha} \xi_{\beta)} \tag{1.74}$$

The contraction of the vector ξ^μ with $\Omega^\mu_{\alpha\beta}$ gives,

$$\xi_\mu \Omega^\mu{}_{\alpha\beta} = -\xi_\mu \lambda_\alpha \lambda_\beta \partial^\mu H + 2\overbrace{\xi_\mu \lambda^\mu}^{0} \lambda_{(\alpha}\partial_{\beta)} H + 2\overbrace{\xi_\mu \lambda^\mu}^{0} \lambda_{(\alpha}\xi_{\beta)} H \tag{1.75}$$
$$\xi_\mu \Omega^\mu{}_{\alpha\beta} = -\xi_\mu \lambda_\alpha \lambda_\beta \partial^\mu H.$$

Also,

$$\xi^\alpha \Omega^\mu{}_{\alpha\beta} = -\overbrace{\xi^\alpha \lambda_\alpha}^{0} \lambda_\beta \partial^\mu H + 2\xi_\alpha \lambda^\mu \lambda_{(\alpha}\partial_{\beta)} H + 2\xi^\alpha \lambda^\mu \lambda_{(\alpha}\xi_{\beta)} H. \tag{1.76}$$

The second part of the equation (1.76) gives $\lambda^\mu \lambda_\beta \xi^\alpha \partial_\alpha H$.
Finally, for the last part of the equation, the result will be $\lambda^\mu \lambda_\beta \xi^\alpha \xi_\alpha H$. Hence, the solution can be reachable,

$$\xi_\mu \Omega^\mu{}_{\alpha\beta} = \lambda^\mu \lambda_\beta (\xi^\alpha \partial_\alpha H + \xi^\alpha \xi_\alpha H). \tag{1.77}$$

Therefore, we can see that the equations are not equal ($\nabla_\mu \xi_\rho \neq \bar{\nabla}_\mu \xi_\rho$)

$$\lambda^\mu \partial_\mu H = 0. \tag{1.78}$$

Also, the Kerr-Schild metric satisfies.

$$\bar{\nabla}_\mu \lambda^\mu \lambda_\beta = 0, \tag{1.79}$$

which is the geodesic equation.

$$\lambda^\mu \bar{\nabla}_\mu \lambda_\beta = 0. \tag{1.80}$$

By using the following identity, we are going to find the Ricci tensor,

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) = [\bar{\nabla}_\mu, \bar{\nabla}_\alpha] H\lambda^\mu \lambda_\beta + \bar{\nabla}_\alpha \bar{\nabla}_\mu (H\lambda^\mu \lambda_\beta) \tag{1.81}$$

By using equation (1.68), one can get a new form

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) = \bar{R}_{\alpha\sigma} H\lambda^\sigma \lambda_\beta + \bar{R}^\sigma_{\mu\alpha\beta} H\lambda^\mu \lambda_\sigma + \bar{\nabla}_\alpha (\lambda^\mu \partial_\mu H \lambda_\beta + H\bar{\nabla}_\mu \lambda^\mu \lambda_\beta + H\lambda^\mu \bar{\nabla}_\mu \lambda_\beta). \tag{1.82}$$

With the help of the equations (1.78), (1.79), (1.80), one can write

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) = \bar{R}_{\alpha\sigma} H\lambda^\sigma \lambda_\beta + \bar{R}^\sigma_{\mu\alpha\beta} H\lambda^\mu \lambda_\sigma. \tag{1.83}$$

Hence, by using $\bar{R}^\mu_{\alpha\nu\beta} = -\frac{1}{l^2}(\delta^\nu_\mu \bar{g}_{\alpha\beta} - \delta^\mu_\beta \bar{g}_{\alpha\nu})$, one can reach a new form of the equation (1.81),

$$\bar{R}_{\alpha\sigma} H\lambda^\sigma \lambda_\beta - \frac{1}{l^2} H\lambda^\mu \lambda_\sigma (\delta^\sigma_\alpha \bar{g}_{\mu\beta} - \delta^\sigma_\mu \bar{g}_{\alpha\beta}). \tag{1.84}$$

The second part of the last equation comes as zero,

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) = -\frac{4}{l^2} H\lambda_\alpha \lambda_\beta. \tag{1.85}$$

Ricci tensor is,

$$R^\rho_\beta = \bar{R}^\rho_\beta - 2H\lambda^\rho \lambda^\alpha \bar{R}_{\alpha\beta} + \bar{g}^{\rho\alpha} \bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta}, \tag{1.86}$$

$$\Omega^\mu{}_{\alpha\beta} = \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) + \bar{\nabla}_\beta (H\lambda^\mu \lambda_\alpha) - \bar{\nabla}^\mu (H\lambda_\alpha \lambda_\beta), \tag{1.87}$$

$$\bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta} = \bar{\nabla}_\mu \bar{\nabla}_\alpha (H\lambda^\mu \lambda_\beta) + \bar{\nabla}_\mu \bar{\nabla}_\beta (H\lambda^\mu \lambda_\alpha) - \bar{\Box}(H\lambda_\alpha \lambda_\beta). \tag{1.88}$$

By using equation (1.85) one can find,

$$\bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta} = -\frac{4}{l^2} H\lambda_\alpha \lambda_\beta - \frac{4}{l^2} H\lambda_\beta \lambda_\alpha - \bar{\Box}(H\lambda_\alpha \lambda_\beta). \tag{1.89}$$

$$\begin{aligned} R^\rho_\beta &= \bar{R}^\rho_\beta - 2H\lambda^\rho \lambda^\alpha \bar{R}_{\alpha\beta} + \bar{g}^{\rho\alpha}(-\frac{8}{l^2} H\lambda_\alpha \lambda_\beta - \bar{\Box}(H\lambda_\alpha \lambda_\beta)) \\ &= \bar{R}^\rho_\beta - 2H\lambda^\rho \lambda^\alpha \bar{R}_{\alpha\beta} + (-\frac{8}{l^2} H\lambda^\rho \lambda_\beta - \bar{g}^{\rho\alpha}(\bar{\Box}(H\lambda_\alpha \lambda_\beta)) \\ &= \bar{R}^\rho_\beta + \frac{6}{l^2} H\lambda^\rho \lambda_\beta - \frac{8}{l^2} H\lambda^\rho \lambda_\beta - (\rho - \frac{2}{l^2} H)\lambda^\rho \lambda_\beta, \\ R^\rho_\beta &= \bar{R}^\rho_\beta + \rho\lambda^\rho \lambda_\beta, \end{aligned} \tag{1.90}$$

where

$$\bar{R}_{\alpha\beta} = -\frac{3}{l^2} \bar{g}_{\alpha\beta}, \tag{1.91}$$

$$\bar{\Box}(H\lambda_\alpha \lambda_\beta) = (-\rho - \frac{2}{l^2} H)\lambda_\alpha \lambda_\beta, \tag{1.92}$$

and

$$R_{\rho\beta} = -\frac{3}{l^2} g_{\rho\beta} + \rho\lambda_\rho \lambda_\beta. \tag{1.93}$$

Now, I am going to show how to prove the equation (1.92),

$$\begin{aligned} \bar{\Box}(H\lambda_\alpha \lambda_\beta) &= \bar{\nabla}^\mu (\bar{\nabla}_\mu H\lambda_\alpha \lambda_\beta + H\bar{\nabla}_\mu \lambda_\alpha \lambda_\beta + H\lambda_\alpha \bar{\nabla}_\mu \lambda_\beta), \\ \bar{\Box}(H\lambda_\alpha \lambda_\beta) &= H\lambda_\alpha \bar{\Box}\lambda_\beta + H\lambda_\beta \bar{\Box}\lambda_\alpha + \bar{\Box}H\lambda_\alpha \lambda_\beta + H\bar{\nabla}^\mu \lambda_\alpha \bar{\nabla}_\mu \lambda_\beta + H\bar{\nabla}_\mu \lambda_\alpha \bar{\nabla}^\mu \lambda_\beta \\ &\quad + 2\partial^\mu H(\lambda_\beta \bar{\nabla}_\mu \lambda_\alpha + \lambda_\alpha \bar{\nabla}_\mu \lambda_\beta), \\ \bar{\Box}(H\lambda_\alpha \lambda_\beta) &= -\frac{6}{l^2} H\lambda_\alpha \lambda_\beta + \lambda_\alpha \lambda_\beta (\frac{H}{2} \xi^\mu \xi_\mu + 2\xi_\mu \partial^\mu H) + \lambda_\alpha \lambda_\beta \bar{\Box} H, \\ &= \big(-\rho - \frac{2}{l^2} H\big)\lambda_\alpha \lambda_\beta, \end{aligned} \tag{1.94}$$

where

$$\rho = -\big(2\xi_\mu \partial^\mu H + \frac{H}{2}\xi^\mu \xi_\mu + \bar{\Box} H - \frac{4}{l^2} H\big) \tag{1.95}$$

The Riemann tensor has the form,

$$R^\mu{}_{\alpha\nu\beta} = \bar{R}^\mu{}_{\alpha\nu\beta} + \bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta \Omega^\mu{}_{\alpha\nu} + \Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha} - \Omega^\mu{}_{\beta\sigma}\Omega^\sigma{}_{\nu\alpha}, \tag{1.96}$$

where $\bar{R}^\mu{}_{\alpha\nu\beta}$ is the Riemann tensor of AdS space-time which can be expressed as

$$\bar{R}^\mu{}_{\alpha\nu\beta} = -\frac{1}{l^2}(\delta^\mu_\nu \bar{g}_{\alpha\beta} - \delta^\mu_\beta \bar{g}_{\alpha\nu}). \tag{1.97}$$

The contraction of the Riemann tensor with two λ^μ vectors has the form,

$$\lambda_\mu \lambda^\nu R^\mu{}_{\alpha\nu\beta} = \lambda_\mu \lambda^\nu \bar{R}^\mu{}_{\alpha\nu\beta}, \tag{1.98}$$

$$\begin{aligned} &\lambda_\mu \lambda^\nu (-\frac{1}{l^2}\delta^\mu_\nu \bar{g}_{\alpha\beta} + \frac{1}{l^2}\delta^\mu_\beta \bar{g}_{\alpha\nu}) \\ &= -\frac{1}{l^2}\overbrace{\lambda_\nu \lambda^\nu}^{=0} \bar{g}_{\alpha\beta} + \frac{1}{l^2}\overbrace{\lambda_\beta \lambda^\nu \bar{g}_{\alpha\nu}}^{\lambda_\alpha \lambda_\beta} = \frac{1}{l^2}\lambda_\alpha \lambda_\beta. \end{aligned} \tag{1.99}$$

Also,

$$\lambda^\alpha \lambda_\nu R^\mu{}_{\alpha\nu\beta} = \lambda^\alpha \lambda_\nu \bar{R}^\mu{}_{\alpha\nu\beta} = -\frac{1}{l^2}\lambda^\mu \lambda_\beta, \tag{1.100}$$

$$\begin{aligned} &\lambda^\alpha \lambda^\nu (-\frac{1}{l^2}\delta^\mu \nu \bar{g}_{\alpha\beta} + \frac{1}{l^2}\delta^\mu_\beta \bar{g}_{\alpha\nu}), \\ &= -\frac{1}{l^2}\lambda^\alpha \lambda^\mu \bar{g}_{\alpha\beta} = -\frac{1}{l^2}\lambda_\beta \lambda^\mu. \end{aligned} \tag{1.101}$$

The scalar curvature of Kerr-Schild metrics is a constant having a value and normalized as

$$\bar{R} = R = -\frac{12}{l^2}. \tag{1.102}$$

This might be known that the AdS-wave and spherical-AdS-wave metrics belong to the class of Kerr-Schild-Kundt.

In addition, the trace-free Ricci tensor is,

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{4} g_{\mu\nu}. \tag{1.103}$$

One can write this as,

$$S = \rho \lambda_\mu \lambda_\nu. \tag{1.104}$$

by defining ρ as in (1.95).

Then, the Riemann tensor can be written as

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + (g_{\mu[\nu}S_{\beta]\alpha} - g_{\alpha[\nu}S_{\beta]\nu}) + \frac{R}{6}g_{\mu[\nu}g_{\beta]\alpha}. \tag{1.105}$$

Lots of properties introduced here will be rediscussed extensively in Appendix A3.

CHAPTER 2

PP-WAVE SOLUTIONS TO HIGHER DERIVATIVE GRAVITY

2.1 Quadratic Gravity

In this section we study the exact pp-waves solutions of quadratic gravity theory [19, 23, 24] [1]. The action of quadratic gravity is given as

$$I = \int d^4x\sqrt{-g}\Big[\frac{1}{\kappa}(R-2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma(R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2)\Big] \tag{2.1}$$

The source free field equations are [25, 26].

$$\begin{gathered}\frac{1}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}) + 2\alpha R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + (2\alpha+\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R \\ +\beta\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} + 2\gamma[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} \\ R_{\mu\sigma\rho\tau}R^{\sigma\rho\tau} - 2R_{\mu\sigma}R_\nu^\sigma - \frac{1}{4}(g_{\mu\nu}R_{\tau\lambda\sigma\rho}^2 - 4R_{\sigma\rho}^2 + R^2)] = 0\end{gathered} \tag{2.2}$$

To find pp-wave solutions of the theory, let us consider the field equations of quadratic gravity for the pp-wave metric.

$$\begin{gathered}\frac{1}{\kappa}[-\lambda_\mu\lambda_\nu\partial^2 H - \frac{1}{2}(\eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu)\overbrace{R}^{=0} + \overbrace{\Lambda_0}^{=0}(\eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu)] + 2\alpha \\ \overbrace{R}^{=0}[-\lambda_\mu\lambda_\nu\partial^2 H - \frac{1}{4}(\eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu)\overbrace{R}^{=0}] + (2\alpha+\beta)[(\eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu)\Box - \nabla_\mu\nabla_\nu]\overbrace{R}^{=0} \\ +\beta\Box[(-\lambda_\mu\lambda_\nu\partial^2 H) - \frac{1}{2}(\eta_{\mu\nu} + 2H\lambda_\mu\lambda_\nu)\overbrace{R}^{=0}] + 2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} \\ +2\gamma[\overbrace{R}^{=0}(-\lambda_\mu\lambda_\nu\partial^2 H) - 2\overbrace{R_{\mu\sigma\nu\rho}R^{\sigma\rho}}^{=0} + \overbrace{R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}}^{=0} - 2\overbrace{R_{\mu\sigma}R_\nu^\sigma}^{=0} \\ -\frac{1}{4}g_{\mu\nu}(R_{\tau\lambda\sigma\rho}^2 - 4\overbrace{R_{\sigma\rho}^2}^{=0} + \overbrace{R^2}^{=0})]\end{gathered} \tag{2.3}$$

[1] In this section, for the details on the quadratic gravity, see [24]

Now, let us compute each term in the field equations by using the relations which are given in the sec.1.2.

1.
$$(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} = \overbrace{R_{\mu\sigma\nu\rho}R^{\sigma\rho}}^{a} - \frac{1}{4}g_{\mu\nu}\overbrace{R_{\sigma\rho}R^{\sigma\rho}}^{b}. \tag{2.4}$$

   (a) $R_{\mu\sigma\nu\rho}R^{\sigma\rho} = 0$ since $R^{\sigma\rho}{}_{\alpha}\lambda^{\sigma}\lambda^{\rho} \longrightarrow \lambda^{\sigma}\lambda^{\rho}R_{\mu\sigma\nu\rho} = 0$,

   (b) $R_{\sigma\rho}R^{\sigma\rho} = 0$ because $R^{\sigma\rho} \propto \lambda^{\sigma}\lambda^{\rho}$ and $R_{\sigma\rho} \propto \lambda_{\sigma}\lambda_{\rho}$. Hence, by using equation (1.28), the answer comes zero. $(\lambda^{\sigma}\lambda_{\sigma} = 0, \lambda^{\rho}\lambda_{\rho} = 0)$.

2. $R_{\mu\sigma}R^{\sigma}{}_{\nu} = (\lambda_{\mu}\lambda_{\sigma}\partial^2 H)(-\lambda^{\sigma}\lambda_{\nu}\partial^2 H) = 0$ since λ is a null vector.

3. By using equation ,$R_{\mu\sigma\rho\tau}R_{\nu}{}^{\sigma\rho\tau} = 0$ One can show this by using the properties (1.28) and (1.29).

$$\begin{aligned}
R_{\mu\sigma\rho\tau}R_{\nu}{}^{\sigma\rho\tau} &= [(\lambda_{\mu}\lambda_{\tau}\partial_{\sigma}\partial_{\rho}H + \lambda_{\sigma}\lambda_{\rho}\partial_{\mu}\partial_{\tau}H - \lambda_{\mu}\lambda_{\rho}\partial_{\sigma}\partial_{\tau}H - \lambda_{\sigma}\lambda_{\tau}\partial_{\mu}\partial_{\rho}H) \\
&\quad (\lambda_{\nu}\lambda^{\tau}\partial^{\sigma}\partial^{\rho}H + \lambda^{\sigma}\lambda^{\rho}\partial_{\nu}\partial^{\tau}H - \lambda_{\nu}\lambda^{\rho}\partial^{\sigma}\partial^{\tau}H - \lambda^{\sigma}\lambda^{\tau}\partial_{\nu}\partial^{\rho}H)] \\
&= \overbrace{\lambda^{\tau}\lambda_{\tau}}^{=0}\partial_{\sigma}\partial_{\rho}H\lambda_{\nu}\lambda_{\mu}\partial^{\sigma}\partial^{\rho}H + \overbrace{\lambda^{\sigma}\partial_{\sigma}H}^{=0}\lambda_{\mu}\lambda_{\tau}\partial_{\rho}\lambda^{\rho}\partial_{\nu}\partial^{\tau}H - \overbrace{\lambda^{\rho}\partial_{\rho}H}^{=0}\lambda_{\mu}\lambda_{\tau}\partial_{\sigma}\lambda_{\nu}\partial^{\sigma}\partial^{\tau}H \\
&- \overbrace{\lambda^{\tau}\lambda_{\tau}}^{=0}\partial_{\sigma}\partial_{\rho}H\lambda^{\sigma}\lambda_{\mu}\partial_{\nu}\partial^{\rho}H + \overbrace{\lambda^{\tau}\partial_{\tau}H}^{=0}\lambda_{\sigma}\lambda_{\rho}\partial_{\mu}\lambda_{\nu}\partial^{\sigma}\partial^{\rho}H + \overbrace{\lambda_{\sigma}\lambda^{\sigma}}^{=0}\lambda_{\rho}\partial_{\mu}\partial_{\tau}H\lambda^{\rho}\partial_{\nu}\partial^{\tau}H \\
&- \overbrace{\lambda^{\rho}\lambda_{\rho}}^{=0}\partial_{\mu}\partial_{\tau}H\lambda_{\nu}\lambda_{\sigma}\partial^{\sigma}\partial^{\tau}H - \overbrace{\lambda_{\sigma}\lambda^{\sigma}}^{=0}\lambda_{\rho}\partial_{\mu}\partial_{\tau}H\lambda^{\tau}\partial_{\nu}\partial^{\rho}H - \overbrace{\lambda^{\tau}\partial_{\tau}H}^{=0}\lambda_{\mu}\lambda_{\rho}\partial_{\sigma}\lambda_{\nu}\partial^{\sigma}\partial^{\rho}H \\
&- \overbrace{\lambda^{\rho}\lambda_{\rho}}^{=0}\partial_{\sigma}\partial_{\tau}H\lambda^{\sigma}\lambda_{\mu}\partial_{\nu}\partial^{\tau}H + \overbrace{\lambda^{\rho}\lambda_{\rho}}^{=0}\partial_{\sigma}\partial_{\tau}H\lambda_{\nu}\lambda_{\mu}\partial^{\sigma}\partial^{\tau}H + \overbrace{\lambda^{\sigma}\partial_{\sigma}H}^{=0}\lambda_{\mu}\lambda_{\rho}\partial_{\tau}H\lambda^{\tau}\partial_{\nu}\partial^{\rho} \\
&- \overbrace{\lambda^{\tau}\lambda_{\tau}}^{=0}\lambda_{\sigma}\partial_{\mu}\partial_{\rho}H\lambda_{\nu}\partial^{\sigma}\partial^{\rho}H - \overbrace{\lambda_{\sigma}\lambda^{\sigma}}^{=0}\lambda_{\tau}\partial_{\mu}\partial_{\rho}H\lambda^{\rho}\partial_{\nu}\partial^{\tau}H + \overbrace{\lambda^{\rho}\partial_{\rho}H}^{=0}\lambda_{\sigma}\lambda_{\tau}\partial_{\mu}\lambda_{\nu}\partial^{\sigma}\partial^{\tau}H \\
&+ \overbrace{\lambda_{\sigma}\lambda^{\sigma}}^{=0}\lambda_{\tau}\partial_{\mu}\partial_{\rho}H\lambda^{\tau}\partial_{\nu}\partial^{\rho}H = 0,
\end{aligned} \tag{2.5}$$

Then, we have,

$$\begin{aligned}
R_{\tau\lambda\sigma\rho}R^{\tau\lambda\sigma\rho} &= (\lambda_{\tau}\lambda_{\rho}\partial_{\lambda}\partial_{\sigma}H + \lambda_{\lambda}\lambda_{\sigma}\partial_{\tau}\partial_{\rho}H - \lambda_{\tau}\lambda_{\sigma}\partial_{\lambda}\partial_{\rho}H - \lambda_{\lambda}\lambda_{\rho}\partial_{\tau}\partial_{\sigma}H) \\
&\quad (\lambda_{\tau}\lambda_{\rho}\partial_{\lambda}\partial_{\sigma}H + \lambda_{\lambda}\lambda_{\sigma}\partial_{\tau}\partial_{\rho}H - \lambda_{\tau}\lambda_{\sigma}\partial_{\lambda}\partial_{\rho}H - \lambda_{\lambda}\lambda_{\rho}\partial_{\tau}\partial_{\sigma}H) = 0.
\end{aligned} \tag{2.6}$$

Field equations can be obtained as

$$(\beta\bar{\Box} + \frac{1}{\kappa})R_{\mu\nu} = 0. \tag{2.7}$$

A pp-wave metric that solves $R_{\mu\nu} = 0$ is a solution of the last equation. Additionally, $R_{\mu\nu} = 0$ is a vacuum field equation.

By using the definition of Ricci tensor for pp-wave spacetimes, field equations can be written as

$$(\Box + \frac{1}{\beta\kappa})\partial^2 H = 0. \tag{2.8}$$

where $\partial^2 = 2\frac{\partial^2}{\partial_u \partial_v} + \hat{\partial}^2$ and takes the following form,

$$(\hat{\partial}^2 - m_\beta^2)\hat{\partial}^2 H = 0 \tag{2.9}$$

where $m_\beta^2 = -\frac{1}{\beta\kappa}$ is the mass of the spin-2 excitation.

To find the explicit pp-wave solutions of the quadratic gravity theory, let us consider the pp-wave metric in null coordinates as

$$ds^2 = 2dudv + 2H(u, x, y)dv^2 + dx^2 + dy^2. \tag{2.10}$$

$$\begin{aligned} \Box H = g^{\mu\nu}\nabla_\mu\nabla_\nu H &= (\eta^{\mu\nu} - 2\lambda^\mu\lambda^\nu H)(\nabla_\mu\nabla_\nu H) \\ &= \eta^{\mu\nu}\partial_\mu\partial_\nu H - \eta^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma H. \end{aligned} \tag{2.11}$$

where $u = \frac{1}{\sqrt{2}}(x - t)$ and $v = \frac{1}{\sqrt{2}}(x + t)$ that are light cone background coordinates. Covariant vector $\lambda_\mu = \delta^u_\mu$ yields a contravariant vector as $\lambda^\mu = \delta^\mu_v$. Then, one can obtain,

$$\lambda_\mu dx^\mu = \delta^u_\mu dx^\mu = du, \qquad \lambda^\mu\partial_\mu H = \delta^\mu_\nu\partial_\mu H = \partial_v H = 0. \tag{2.12}$$

By using, equation (2.12) and Laplacian for the metric as $\partial^2 = 2\frac{\partial^2}{\partial u \partial v} + \partial_\perp^2$, $\partial_\perp^2 = \partial_x^2 + \partial_y^2$. Also, (2.8) takes the following form [19],

$$(\partial_\perp^2 - m_\beta^2)\partial_\perp^2 H = 0. \tag{2.13}$$

2.2 Infinite Derivative Gravity

PP-wave space-times are exact solutions of the IDG. We also represent that these waves solve not only non-linear field equations but also the linearized field equations. Now, we will briefly review the IDG. Also, we will give some basic concepts of the pp-wave space-times and represent that these space-times are exact solution of the theory.

2.2.1 The Field Equations of IDG

The most general infinite derivative action in four dimension, around constant curvature backgrounds, parity-invariant, metric-compatible and torsion-free can be expressed as [1–3]

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R+\alpha_c(R\ \ F_1(\Box)R+R_{\mu\nu}F_2(\Box)R^{\mu\nu}+C_{\mu\nu\rho\sigma}F_3(\Box)C^{\mu\nu\rho\sigma})\Big]. \tag{2.14}$$

with

$$F_i(\Box) = \sum_{n=0}^{\infty} f_{i_n}\frac{\Box^n}{M_s^{2n}} \tag{2.15}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. $F_i(\Box)$ contains infinite derivative functions, f_{i_n} is a dimensionless coefficients and avoid ghost-like instabilities. In the limit $\alpha_c \longrightarrow 0$, theory reduces to Einstein's gravity with a massless spin 2 graviton. Also, each $\Box$ term comes with mass scale M^2 where $M < M_p = [16\pi G]^{-\frac{1}{2}}$.
The source-free field equations are [4]

$$\begin{gathered}
G^{\alpha\beta} + \frac{\alpha_c}{2}[4G^{\alpha\beta}\ \ F_1(\Box)\ \ R + g^{\alpha\beta}\ \ R\ \ F_1(\Box)\ \ R - 4\ \ (\nabla^\alpha\nabla^\beta - g^{\alpha\beta}\Box)\ \ F_1\ \ (\Box)\ \ R \\
+4R^\alpha{}_\nu\ \ F_2(\Box)\ \ R^{\nu\beta} - g^{\alpha\beta}\ \ R_\nu{}^\mu F_2(\Box)\ \ R_\mu{}^\nu - 4\ \ \nabla_\nu\nabla^\beta\ \ (F_2(\Box)\ \ R^{\nu\alpha}) + 2\Box(F_2(\Box)\ \ R^{\alpha\beta}) \\
+2\ \ g^{\alpha\beta}\nabla_\mu\nabla_\nu\ \ (F_2(\Box)\ \ R^{\mu\nu}) - g^{\alpha\beta}\ \ C^{\mu\nu\rho\sigma}\ \ F_3(\Box)\ \ C_{\mu\nu\rho\sigma} + 4C^\alpha{}_{\mu\nu\sigma}\ \ F_3(\Box)\ \ C^{\beta\mu\nu\sigma} \\
-4\ \ (R_{\mu\nu} + 2\nabla_\mu\nabla_\nu)\ \ (F_3(\Box)\ \ C^{\beta\mu\nu\alpha)} - 2\ \ \Omega_1^{\alpha\beta} + g^{\alpha\beta}\ \ (\Omega_{1\rho}^\rho + \bar{\Omega}_1) - 2\ \ \Omega_2^{\alpha\beta} \\
+g^{\alpha\beta}(\Omega_{2\rho}^\rho + \bar{\Omega}_2) - 4\Delta_2^{\alpha\beta} - 2\ \ \Omega_3^{\alpha\beta} + g^{\alpha\beta}\ \ (\Omega_{3\gamma}^\gamma + \bar{\Omega}_3) - 8\ \ \Delta_3^{\alpha\beta}] = 0.
\end{gathered} \tag{2.16}$$

Here, the symmetric tensors are given as,

$$\Omega_1^{\alpha\beta} = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \nabla^\alpha R^l \nabla^\beta R^{(n-l-1)}, \quad \bar{\Omega}_1 = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} R^l R^{(n-l)},$$

$$\Omega_2^{\alpha\beta} = \sum_{n=1}^{\infty} f_{2n} R_\nu^{\mu;\alpha(l)} R_\mu^{\nu;\beta(n-l-1)}, \quad \bar{\Omega}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} R_\nu^{\mu(l)} R_\mu^{\nu(n-l)},$$

$$\triangle_2^{\alpha\beta} = \frac{1}{2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} [R_\sigma^{\nu(l)} R^{(\beta|\sigma|;\alpha)(n-l-1)} - R_\sigma^{\nu;(\alpha(l)} R^{\beta)\sigma(n-l-1)}]_{;\nu},$$

$$\Omega_3^{\alpha\beta} = \frac{1}{2} \sum_{n-1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\nu\lambda\sigma}^{\mu;\alpha(l)} C_\mu^{\nu\lambda\sigma;\beta(n-l-1)}, \quad \bar{\Omega}_3 = \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\nu\lambda\sigma}^{\mu(l)} C_\mu^{\nu\lambda\sigma(n-l)},$$

$$\triangle_3^{\alpha\beta} = \frac{1}{2} \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} [C_{\sigma\mu}^{\lambda\nu(l)} C_\lambda^{(\beta|\sigma\mu|;\alpha)(n-l-1)} - C_{\sigma\mu}^{\lambda\nu;(\alpha(l)} C_\lambda^{\beta)\sigma\mu(n-l-1)}]_{;\nu}. \tag{2.17}$$

Now, let us express the field equations for the pp-wave space-times. Hence, the field equations for the pp-wave space-times take the following form,

$$\begin{aligned}
&G^{\alpha\beta} + \tfrac{\alpha_c}{2} [\overbrace{4R^\alpha{}_\nu \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}} (-\lambda^\nu \lambda^\beta \partial^2 H)}^{1} - \overbrace{g^{\alpha\beta} R_\nu{}^\mu \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}} R_\mu{}^\nu}^{2} - 4 \overbrace{\nabla_\nu \nabla^\beta \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}} (-\lambda^\nu \lambda^\alpha \partial^2 H)}^{3} \\
&+ 2 \overbrace{\Box \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}} (-\lambda^\alpha \lambda^\beta \partial^2 H)}^{4} + 2 \overbrace{g^{\alpha\beta} \nabla_\mu \nabla_\nu \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}} (-\lambda^\mu \lambda^\nu \partial^2 H)}^{5} - \overbrace{g^{\alpha\beta} C^{\mu\nu\rho\sigma} (\sum_{n=0}^{\infty} f_{3n} \frac{\Box^n}{M^{2n}}) C_{\mu\nu\rho\sigma}}^{6} \\
&+ \overbrace{\sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} R_\nu{}^{\mu(l)} R_\mu{}^{\nu(n-l)}}^{12}) - 2 (\overbrace{+ \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} [R_\sigma{}^{\nu(l)} R^{(\beta|\sigma|;\alpha)(n-l-1)} - R_\sigma^{\nu;(\alpha(l)} R^{\beta)\sigma(n-l-1)}]}^{13}; \nu \\
&- 2 \overbrace{\sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\nu\rho\sigma}^{\mu;\alpha(l)} C_\mu{}^{\nu\rho\sigma;\beta(n-l-1)}}^{14} + \overbrace{g^{\alpha\beta} (\Omega_{3\gamma}^\gamma + \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\nu\rho\sigma}^{\mu(l)} C_\mu^{\nu\rho\sigma(n-l)})}^{15} \\
&- 4 \overbrace{\sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} [C^{\rho\nu(l)}{}_{\sigma\mu} C_\rho{}^{(\beta|\sigma\mu|;\alpha)(n-l-1)} - C^{\rho\nu}{}_{\sigma\mu}{}^{;(\alpha(l)} C_\rho^{\beta)\sigma\mu(n-l-1)}; \nu]}^{16} = 0
\end{aligned} \tag{2.18}$$

where we used the fact that $R = 0$. Recall that the Weyl tensor for the pp-wave space-times can be described as

$$\begin{aligned}
C_{\mu\nu\rho\sigma} = &(\lambda_\mu \lambda_\sigma \partial_\nu \partial_\rho H + \lambda_\nu \lambda_\rho \partial_\mu \partial_\sigma H - \lambda_\mu \lambda_\rho \partial_\nu \partial_\rho H - \lambda_\nu \lambda_\sigma \partial_\mu \partial_\rho H \\
&- \frac{1}{2} (\eta_{\mu\sigma} \lambda_\rho \lambda_\nu + \eta_{\nu\rho} \lambda_\sigma \lambda_\mu - \eta_{\mu\rho} \lambda_\sigma \lambda_\nu - \eta_{\nu\sigma} \lambda_\rho \lambda_\mu) \partial^2 H.
\end{aligned} \tag{2.19}$$

Let us give as an example, calculations of the some parts,

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = R^{\alpha\beta}, \tag{2.20}$$

where $R=0$.

$$(1) \longrightarrow R^{\alpha}{}_{\nu}F_2(\Box)R^{\nu\beta} = R^{\alpha}{}_{\nu}\lambda^{\nu}\lambda^{\beta}\sum_{n=0}^{\infty} f_{2n}\frac{\Box^n}{M^{2n}}(-\partial^2 H) = 0,$$

where

$$\lambda^{\nu}R_{\alpha\nu} = 0,$$

$$(2) \longrightarrow g^{\alpha\beta}R_{\nu}{}^{\mu}\sum_{n=0}^{\infty} f_{2n}\frac{\Box^n}{M^{2n}}R_{\mu}{}^{\nu} = g^{\alpha\beta}(-\lambda_{\nu}\lambda^{\mu}\partial^2 H)\sum_{n=0}^{\infty} f_{2n}\frac{\Box^n}{M^{2n}}(-\lambda_{\mu}\lambda^{\nu}\partial^2 H) = 0,$$

By using $\lambda^{\mu}\lambda_{\mu} = 0$.

$$(3) \longrightarrow \nabla_{\nu}\nabla^{\beta}\sum_{n=0}^{\infty} f_{2n}\frac{\Box^n}{M^{2n}}(-\lambda^{\nu}\lambda^{\alpha}\partial^2 H) = 0.$$

By using a property which is $\nabla_{\mu}\lambda_{\nu} = 0$.

$(4) \longrightarrow 2\Box(F_2(\Box)R^{\alpha\beta}) = 2\Box\sum_{n=0}^{\infty} f_{2n}\frac{\Box^n}{M^{2n}}(-\lambda^{\alpha}\lambda^{\beta}\partial^2 H)$.

$(5) \longrightarrow$ by using a property which is $\nabla_{\mu}\lambda_{\nu} = 0$.

$(6), (7) \longrightarrow$ By using $\lambda^{\mu}\lambda_{\mu} = 0$. This property is satisfied by Weyl tensor.

$(8) \longrightarrow (R_{\mu\nu}+2\nabla_{\mu}\nabla_{\nu})(F_3(\Box)C^{\beta\mu\nu\alpha}) = (-\lambda_{\mu}\lambda_{\nu}\partial^2 H)F_3(\Box)C^{\beta\mu\nu\alpha}+2\nabla_{\mu}\nabla_{\nu}F_3(\Box)C^{\beta\mu\nu\alpha}$

$$= -\overbrace{\lambda_{\mu}\lambda_{\nu}\sum_{n=0}^{\infty} f_{3n}\frac{\Box^n}{M^{2n}}C^{\beta\mu\nu\alpha}}^{=0} + F_3(\Box)\nabla_{\mu}\nabla_{\nu}\overbrace{C^{\beta\mu\nu\alpha}}^{-C^{\mu\beta\nu\alpha}} = -\tfrac{1}{2}F_3(\Box)\Box R^{\alpha\beta}$$

where we used the following relation

$$\nabla_{\alpha}\nabla_{\beta}C^{\beta\mu\alpha\nu} = \frac{1}{2}\Box R^{\mu\nu} \tag{2.21}$$

The second part of (8) is $\nabla_{\mu}\nabla_{\nu}F_3(\Box)C^{\beta\mu\nu\alpha}$.

$(9), (10) \longrightarrow \Omega_1^{\alpha\beta} = \bar{\Omega}_1 = 0$ because R=0.

$(12) \longrightarrow \bar{\Omega}_2 = \sum_{n=1}^{\infty} f_{2n}\sum_{l=0}^{n-1} R_{\nu}{}^{\mu(l)}R_{\mu}{}^{\nu(n-l)} = \sum_{n=1}^{\infty} f_{2n}\sum_{l=0}^{n-1}\Box^l R_{\nu}{}^{\mu}\Box^{n-l}R_{\mu}{}^{\nu}$

$$= \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \Box^l(-\lambda_\nu \lambda^\mu \partial^2 H)\Box^{n-l}(-\lambda_\mu \lambda^\nu \partial^2 H) = 0,$$

Now, one can see that the field equations take a new form,

$$\begin{aligned} &[R^{\alpha\beta} + \alpha_c(\Box F_2(\Box)R^{\alpha\beta} + 2F_3(\Box)\Box R^{\alpha\beta}] \\ &= [1 + \alpha_c(\Box F_2(\Box) + 2F_3(\Box)\Box)]R^{\alpha\beta} = 0. \end{aligned} \tag{2.22}$$

It is easy to see the field equations in this form,

$$[1 + \alpha_c(\Box F_2(\Box) + 2F_3(\Box)\Box)]R_{\mu\nu} = 0. \tag{2.23}$$

By using Ricci tensor definition, the field equations can be rearranged as

$$[1 + \alpha_c(\Box F_2(\Box) + 2F_3(\Box)\Box)]\partial^2 H = 0. \tag{2.24}$$

Form factors in the equation (2.24) can be described as

$$F_2(\Box) = \sum_{n=0}^{\infty} f_{2n} \frac{\Box^n}{M^{2n}}, \qquad F_3(\Box) = \sum_{n=0}^{\infty} f_{3n} \frac{\Box^n}{M^{2n}}. \tag{2.25}$$

Consider the box operator on acting on H to get $\Box^n H$. By using equation (2.11), one can arrange $\Box H$ as,

$$\Box H = g^{\mu\nu}\nabla_\mu \nabla_\nu H = \eta^{\mu\nu}\partial_\mu \partial_\nu H - \eta^{\mu\nu}\Gamma^\sigma_{\mu\nu}\partial_\sigma H \tag{2.26}$$

One can show that the second terms in (2.26) will vanish,

$$\eta^{\mu\nu}[\lambda^\sigma \lambda_\mu \partial_\nu H + \lambda^\sigma \lambda_\nu \partial_\mu H - \lambda_\mu \lambda_\nu \eta^{\sigma\beta}\partial_\beta H] = \lambda^\sigma \overbrace{\lambda^\nu \partial_\nu H}^{=0} + \lambda^\sigma \overbrace{\lambda^\mu \partial_\mu H}^{=0} - \overbrace{\lambda^\nu \lambda_\nu}^{=0} \eta^{\sigma\beta}\partial_\beta H = 0 \tag{2.27}$$

where $\lambda^\nu \lambda_\nu = 0$ and $\lambda^\mu \partial_\mu H = 0$. Then, the equation (2.26) can be written as

$$\Box H = \partial^2 H, \tag{2.28}$$

To reduce the field equations of IDG (2.24), one can prove that $\Box^n \partial^2 H = \partial^{2n}(\partial^2 H)$,

$$\Box^n \partial^2 H = \Box^n(\partial^2 H) = \partial^2 \overbrace{(\Box^n H)}^{\partial^{2n} H} = \partial^{2n}(\partial^2 H). \tag{2.29}$$

Hence, the field equations of IDG takes the form [8],

$$[1 + \alpha_c(\partial^2 F_2(\partial^2) + 2\partial^2 F_3(\partial^2))]\partial^2 H = 0. \tag{2.30}$$

2.3 PP-Wave Solutions

In this section , we will construct the exact pp-wave solutions of infinite derivative gravity [8]. One can show this,

$$\Box H = \partial^2 H = (2\frac{\partial^2}{\partial u \partial v} + \partial_\perp^2)H = 2\frac{\partial^2 H}{\partial u \partial v} + \partial_\perp^2 H = \partial_\perp^2 H, \tag{2.31}$$

where $\partial_v H = 0$. Hence, the equation (2.30) takes the form,

$$[1 + \alpha_c(\partial_\perp^2 F_2(\partial_\perp^2) + 2\partial_\perp^2 F_3(\partial_\perp^2))]\partial_\perp^2 H = 0, \tag{2.32}$$

Form factors can be chosen as [2, 3],

$$F_2(\Box) = \frac{-1 + e^{-\frac{\Box}{M^2}}}{\frac{\Box}{M^2}}, \qquad F_3(\Box) = 0. \tag{2.33}$$

With the choice, theory has no ghosts or extra degrees of freedom other than massless spin-2 degrees of freedom. The corresponding field equation (2.32) can be found,

$$\begin{gathered}
\left(1 + \alpha_c\Big[\partial_\perp^2(\frac{-1 + e^{-\frac{\Box}{M^2}}}{\frac{\partial_\perp^2}{M^2}})\Big]\right)\partial_\perp^2 H = 0, \\
\left(1 + \alpha_c\Big[(\frac{-1 + e^{-\frac{\Box}{M^2}}}{\frac{1}{M^2}})\Big]\right)\partial_\perp^2 H = 0 \\
\left(1 - \alpha_c M^2 + \alpha_c\big(e^{-\frac{\partial_\perp^2}{M^2}}\big)M^2\right)\partial_\perp^2 H = 0, \\
e^{-\frac{\partial_\perp^2}{M^2}}\partial_\perp^2 H = 0,
\end{gathered} \tag{2.34}$$

where $\alpha_c = \frac{1}{M^2}$.

Equation could be solved by using the new method which is known as eigenvalue-method [27],

$$\partial_\perp^2 H = -\alpha^2 H, \tag{2.35}$$

where H are eigenfunctions and α are eigenvalues. To reach the last field equations' form, acting on H,

$$e^{-\frac{\partial_\perp^2}{M^2}}\partial_\perp^2 H = e^{\frac{\alpha^2}{M^2}}\alpha^2\partial_\perp^2 H. \tag{2.36}$$

One can see that the equation reduces to new form,

$$\partial_\perp^2 V = 0. \tag{2.37}$$

Solutions of the source free theory are same with the Einstein's GR. To see the other effects and different solutions, one can consider the field equations in the presence of a source.

2.3.1 Shock Wave Solution of IDG

In this part, we are going to examine the PP-wave solutions in the existence of the radiation sources [8] [2]. Gravitational shock-wave solution will explain the gravitational interactions between high energy massless particles in IDG. Shock waves' metric produced by a moving massless point particle can be written as follows,

$$ds^2 = -dudv + \delta(u)g(x_\perp)du^2 + dx_\perp^2, \tag{2.38}$$

where $u = t - z$ and $v = t + z$ are the transverse coordinates to wave propagation and $g(x_\perp)$ is the wave profile function. To obtain the solution of the exact shock wave of IDG, we will find the form of the wave profile function $g(x_\perp)$. Let us consider the massless point particle travels in the positive z direction with momentum $p^\mu = |p|(\delta_t^\mu + \delta_z^\mu)$. The related null source creates the shock-wave geometry can be written as $T_{uu} = |p|\delta(x_\perp)\delta(u)$. Let us introduce the Ricci tensor for the shock waves,

$$R_{uu} = -\frac{\delta(u)}{2}\frac{\partial^2}{\partial_\perp^2}g(x_\perp). \tag{2.39}$$

For the Kerr-Schild form, the energy-momentum tensor can be defined as $T_{\mu\nu} = |p|\delta(x_\perp)\delta(u)\lambda_\mu\lambda_\nu$. The null source coupled IDG field equations can be recast as

$$[1 + \alpha_c(\partial_\perp^2 F_2(\partial_\perp^2) + 2\partial_\perp^2 F_3(\partial_\perp^2))]\partial_\perp^2 g(x_\perp) = -16\pi G|p|\delta(x_\perp). \tag{2.40}$$

For the form factors, equation (2.40) becomes a modified Poisson equation,

$$e^{-\frac{\partial_\perp^2}{M^2}}\partial_\perp^2 g(x_\perp) = -16\pi G|p|\delta(x_\perp). \tag{2.41}$$

By using Fourier transform a solution can be obtained, one can calculate this step by step,

$$e^{-\frac{\partial_\perp^2}{M^2}}\partial_\perp^2 g = -\kappa\delta(x_\perp), \tag{2.42}$$

[2] In this subsection, or the details, we followed this article [8]

where $\kappa = 16\pi G|p|$. By Fourier transformation,

$$\tilde{g} = \frac{\kappa}{2\pi}\frac{e^{-\frac{p^2}{M^2}}}{p^2}."\tag{2.43}$$

Then,

$$\begin{aligned} g &= (\frac{\kappa}{2\pi})(\frac{1}{2\pi})\int\int \frac{e^{-\frac{p^2}{M^2}}.e^{-i\vec{p}.\vec{r}}}{p^2}pdpd\theta \\ &= \frac{\kappa}{4\pi^2}\int_0^\infty \frac{e^{-\frac{p^2}{M^2}}}{p}dp\overbrace{\int e^{-iprcos\theta}d\theta}^{2\pi J_0(pr)} \end{aligned}\tag{2.44}$$

where $\vec{p}.\vec{r} = prcos\theta$.

The solution reduces to,

$$g(r) = \frac{\kappa}{2\pi}\int_0^\infty \frac{e^{-\frac{p^2}{M^2}}}{p}J_0(pr)dp.\tag{2.45}$$

By using $J_0'(x) = J_1(x)$. The equation reduces to

$$\frac{dg}{dr} = \int e^{-\frac{p^2}{M^2}}J_1(pr)dp.\tag{2.46}$$

$$g(r) = -8G|p|ln\big(\frac{r}{r_0} - \frac{1}{2}E_i(-\frac{r^2M_s^2}{4})\big)\tag{2.47}$$

The equation reduces to a new form (modified Poisson type equation) which is (2.41)

The profile function becomes by using $M_s \longrightarrow \infty$ limit,

$$g(r) = -8G|p|ln(\frac{r}{r_0}),\tag{2.48}$$

which is the Einstein's gravity result which was expected. Therefore, the gravitational shock-wave solution metric for IDG is

$$ds^2 = -dudv - 4G|p|\delta(u)\left(ln(\frac{r^2}{r_0^2} - Ei(\frac{-r^2M_s^2}{4})\right)du^2 + dx_\perp^2.\tag{2.49}$$

where Ei is the exponential integral function.

CHAPTER 3

ADS PLANE WAVES IN HIGHER DERIVATIVE GRAVITY

3.1 AdS Plane Wave Solutions of Quadratic Gravity

Quadratic gravity played a crucial role in constructing the solutions of the generic gravity theory [1]. The field equations of quadratic gravity are given and for AdS-plane waves metric reduce to a trace part and nonlinear wave types of equation on the traceless Ricci tensor (3.24). In this section, we study the exact solutions with the help of the Chapter 3. To obtain the field equations for PP-waves in this theory with $\Lambda_0 = 0$, one simply takes the $l \longrightarrow \infty$ limit. Note that in this limit $S_{\mu\nu} = R_{\mu\nu}$.
In the previous section, it was argued that the metric in the form of (1.53) gives a detail about the relation between field equations of quadratic gravity and solutions of the linearized field equations.AdS-plane waves and AdS-spherical waves of quadratic gravity theories played an important role. We will study here the AdS-plane wave given as

$$ds^2 = \frac{l^2}{z^2}(2dudv + dx.dx + dz^2) + 2V(u, x, z)du^2, \tag{3.1}$$

where u and v are both null coordinates and l is the AdS radius.
The traceless Ricci tensor is

$$S = \rho\lambda_\mu\lambda_\nu. \tag{3.2}$$

For the class of Kerr-Schild-Kundt metric and for D-dimensional metric, the Ricci tensor is

$$R_{\mu\nu} = -\frac{3}{l^2}g_{\mu\nu} + \rho\lambda_\mu\lambda_\nu \tag{3.3}$$

where

$$\rho \equiv -\left(\Box + 2\xi^\mu\partial_\mu + \frac{1}{2}\xi^\mu\xi_\mu - \frac{4}{l^2}\right)H, \tag{3.4}$$

[1] In this chapter, for the details on the AdS plane waves in higher derivative gravity, see [18].

which is the scalar function.

The Ricci scalar is

$$R = -\frac{12}{l^2}. \tag{3.5}$$

$C_{\mu\alpha\nu\beta}$ in terms of the full metric quantities can be written as

$$\begin{aligned} C_{\mu\alpha\nu\beta} = \lambda_\mu\lambda_\nu\Big[&- \nabla_\alpha\partial_\beta H - \xi_{(\alpha}\partial_{\beta)}H - \frac{1}{2}\xi_\alpha\xi_\beta H - \frac{1}{2}g_{\alpha\beta}\Big(\rho - \frac{4}{l^2}H\Big)\Big] \\ +\lambda_\alpha\lambda_\beta\Big[&- \nabla_\mu\partial_\nu H - \xi_{(\mu}\partial_{\nu)}H - \frac{1}{2}\xi_\mu\xi_\nu H - \frac{1}{2}g_{\mu\nu}\Big(\rho - \frac{4}{l^2}H\Big)\Big] \\ -\lambda_\mu\lambda_\beta\Big[&- \nabla_\alpha\partial_\nu H - \xi_{(\alpha}\partial_{\nu)}H - \frac{1}{2}\xi_\alpha\xi_\nu H - \frac{1}{2}g_{\mu\nu}\Big(\rho - \frac{4}{l^2}H\Big)\Big] \\ -\lambda_\alpha\lambda_\nu\Big[&- \nabla_\mu\partial_\beta H - \xi_{(\mu}\partial_{\beta)}H - \frac{1}{2}\xi_\mu\xi_\beta H - \frac{1}{2}g_{\mu\beta}\Big(\rho - \frac{4}{l^2}H\Big)\Big] \end{aligned} \tag{3.6}$$

Then, the definition

$$\Omega_{\alpha\beta} = -\Big[\nabla_\alpha\partial_\beta H + \xi_{(\alpha}\partial_{\beta)} + \frac{1}{2}\xi_\alpha\xi_\beta H + \frac{1}{2}g_{\alpha\beta}\Big(\rho - \frac{4)}{l^2}H\Big)\Big] \tag{3.7}$$

The Weyl in tensor in the general form is

$$C_{\mu\alpha\nu\beta} = 4\lambda_{[\mu}\Omega_{\alpha][\beta}\lambda_{\nu]} \tag{3.8}$$

3.1.1 The Field Equations of Quadratic Gravity

If we rearrange the equation (2.2) [23], by using new properties like $\alpha = 0$, $\beta = 0$, $\gamma = 0$,

$$\frac{1}{\kappa}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}] = 0, \tag{3.9}$$

where

$$R_{\mu\nu} = -\frac{3}{l^2}g_{\mu\nu} + \lambda_\mu\lambda_\nu\rho, \tag{3.10}$$

where

$$R = g^{\mu\nu}R_{\mu\nu} = -\frac{3}{l^2}g^{\mu\nu}g_{\mu\nu}. \tag{3.11}$$

One can see easily that this term is valid for four dimensions $(D = 4)$.

$$\mathcal{O} = -[\bar{\Box} + \frac{4z}{l^2}\partial_z - \frac{2}{l^2}]H, \tag{3.12}$$

$$\Lambda_0 = 0. \tag{3.13}$$

Hence,

$$\frac{1}{\kappa}[-\lambda_\mu\lambda_\nu(\bar{\Box}+\frac{4z}{l^2}\partial_z-\frac{2}{l^2})]H=0, \tag{3.14}$$

which means that H satisfies the $\rho=0$ equation.

The equation (2.2) is the field equations of quadratic gravity, where

$$R=-\frac{12}{l^2}, \tag{3.15}$$

$$R_{\mu\nu}=-\frac{3}{l^2}g_{\mu\nu}+\rho\lambda_\mu\lambda_\nu, \tag{3.16}$$

$$\begin{aligned} S_{\mu\nu}&=R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\Big(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\Big),\\ S_{\mu\nu}&=-\frac{3}{l^2}g_{\mu\nu}+\rho\lambda_\mu\lambda_\nu-\frac{1}{4}g_{\mu\nu}(-\frac{12}{l^2}),\\ S_{\mu\nu}&=\rho\lambda_\mu\lambda_\nu. \end{aligned} \tag{3.17}$$

For (3.1), the equation can be rearranged as,

$$\begin{aligned} &\frac{1}{\kappa}\Big[-\frac{3}{l^2}g_{\mu\nu}+\frac{6}{l^2}g_{\mu\nu}+\Lambda_0 g_{\mu\nu}\Big]+2\alpha\Big[-\frac{12}{l^2}\big(-\frac{3}{l^2}g_{\mu\nu}-S_{\mu\nu}\big)\Big]\\ &-\frac{72}{l^4}\alpha g_{\mu\nu}+(2\alpha+\beta)\Big[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\Big]\Big(-\frac{12}{l^2}\Big)\\ &+\beta\Box\Big[-\frac{3}{l^2}g_{\mu\nu}-S_{\mu\nu}-\frac{1}{2}g_{\mu\nu}-\frac{12}{l^2}\Big]+2\beta\Big[R_{\mu\sigma\nu\rho}R^{\sigma\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho}\Big]\\ &+2\gamma\Big[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}-2R_{\mu\sigma}R_\nu{}^{\sigma}-\frac{1}{4}g_{\mu\nu}R^2_{\tau\lambda\sigma\rho}\\ &-g_{\mu\nu}R^2_{\sigma\rho}-\frac{1}{4}g_{\mu\nu}R^2\Big]=0. \end{aligned} \tag{3.18}$$

By doing the term substitutions and by seperating the terms, I can obtain the next one,

$$\begin{aligned} &g_{\mu\nu}\Big[\frac{3}{\kappa l^2}+\frac{\Lambda}{\kappa}\Big]+g_{\mu\nu}\Big[-\frac{24\alpha\Box}{l^2}-\frac{15\beta\Box}{l^2}-\frac{\beta\Box}{2}+\frac{18\beta}{l^4}-\frac{9}{l^4}\Big]+g_{\mu\nu}\big(-\frac{84}{l^4}\big)\\ &+g_{\mu\nu}\big(-\frac{216\gamma}{l^4}\big)+S_{\mu\nu}\Big[-\frac{1}{\kappa}+\frac{24}{l^2}-\beta\Box\Big]+S_{\mu\nu}[\frac{4\beta}{l^4}-\frac{40\gamma}{l^2}]\\ &+\frac{24\alpha\nabla_\mu\nabla_\nu}{l^2}+\frac{12\beta\nabla_\mu\nabla_\nu}{l^2}-\frac{12\beta\Box}{l^2} \end{aligned} \tag{3.19}$$

where

$$R_{\mu\sigma\nu\rho}R^{\sigma\rho}=\frac{9}{l^4}g_{\mu\nu}+\frac{2}{l^2}S_{\mu\nu}, \tag{3.20}$$

$$RR_{\mu\nu}=\frac{12}{l^2}[\frac{3}{l^2}g_{\mu\nu}+S_{\mu\nu}], \tag{3.21}$$

$$R_{\mu\sigma\rho\tau}R_{\nu}{}^{\sigma\rho\tau} = \frac{6}{l^4}g_{\mu\nu} + 4\frac{S_{\mu\nu}}{l^2}. \tag{3.22}$$

Then, the new form of the equation is,

$$\begin{aligned} g_{\mu\nu}\Big[\frac{\Lambda_0}{\kappa} + \frac{3}{\kappa l^2} - \frac{9}{l^4} + \frac{144}{l^4}\Big(-\frac{\alpha}{2} - 2\gamma\Big) + \frac{18}{l^4}\Big(\beta - 2\gamma\Big) - \gamma\frac{48}{l^4}\Big] \\ +S_{\mu\nu}\Big[-\frac{1}{\kappa} + \frac{24}{l^2}\Big(\alpha - \gamma\Big) + \frac{4\beta}{l^2} - \frac{16\gamma}{l^2}\Big] \\ +\frac{12}{l^2}\Big[2\alpha\nabla_\mu\nabla_\nu + \beta\nabla_\mu\nabla_\nu - \beta\Box\Big] = 0. \end{aligned} \tag{3.23}$$

Hence, one can arrange and find this last form of this equation,

$$\Big(\frac{\Lambda_0}{\kappa} + \frac{3}{\kappa l^2} - f\frac{18}{l^4}\Big)g_{\mu\nu} + \beta\Big(\Box + \frac{2}{l^2} - M^2\Big)S_{\mu\nu} = 0, \tag{3.24}$$

where

$$M^2 = -\frac{1}{\beta}\Big[\frac{1}{\kappa} - \frac{2}{l^2}\Big(12\alpha + 3\beta\Big)\Big]. \tag{3.25}$$

$$f = 0 \tag{3.26}$$

The trace part of the (3.24) gives

$$\frac{\Lambda_0}{\kappa} + \frac{3}{\kappa l^2} - f\frac{18}{l^4} = 0 \tag{3.27}$$

which also determines the cosmological constant.

The traceless part of the field equation reads

$$(\bar{\Box} + \frac{2}{l^2} - M^2)(\bar{\Box} + \frac{2}{l^2})(\lambda_\mu\lambda_\nu H) = 0, \tag{3.28}$$

The metric function H satisfies a fourth-order equation,

$$\Big(\Box + \frac{4z}{l^2}\partial_z - \frac{2}{l^2} - M^2\Big) \times \Big(\Box + \frac{4z}{l^2}\partial_z - \frac{2}{l^2}\Big)H(u, \vec{x}, z) = 0, \tag{3.29}$$

the general solution of (3.29) can be written from the second-order parts; first one is the pure Einstein theory,

$$\Big(\Box + \frac{4z}{l^2}\partial_z - \frac{2}{l^2}\Big)H_a(u, \vec{x}, z) = 0, \tag{3.30}$$

and the other is a "massive" version of the theory

$$\Big(\Box + \frac{4z}{l^2}\partial_z - \frac{2}{l^2}\Big)H_b(u, \vec{x}, z) = 0. \tag{3.31}$$

with $H = H_a + H_b$. I know H_a, let us try to write H_b [18],

$$H_b(u, \vec{x}, z) = z^{\frac{D-5}{2}}[a_{b,1}I_{\nu_b}(z\xi_b) + a_{b_2}K_{v_b}(z\xi_b)] \times sin(\vec{\xi_b}.\vec{x} + a_{b_3}). \tag{3.32}$$

where $\nu_b = \frac{1}{2}\sqrt{(D-1)^2 + 4l^2M^2}$.

3.2 Infinite Derivative Gravity

In the Kerr-Schild form,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2H\lambda_\mu\lambda_\nu, \tag{3.33}$$

where $\bar{g}_{\mu\nu}$ is an AdS background metric. H is a scalar function that satisfies

$$\lambda^\mu \partial_\mu H = 0. \tag{3.34}$$

λ_μ is a vector satisfying,

$$\lambda^\mu\lambda_\mu = 0, \qquad \nabla_\mu\lambda_\nu = \xi(_\mu\lambda_\nu), \qquad \xi_\mu\lambda^\mu = 0 \tag{3.35}$$

The curvature scalar R is constant. Additionally, the Ricci tensor, becomes

$$R_{\mu\nu} = -\frac{3}{l^2}g_{\mu\nu} + \lambda_\mu\lambda_\nu\mathcal{O}H, \tag{3.36}$$

where l is the AdS radius as well as $\mathcal{O}$ defines the operator reads,

$$\mathcal{O} = -\Big(\Box + 2\ \ \xi^\mu\partial_\mu + \frac{1}{2}\xi^\mu\xi_\mu - \frac{4}{l^2}\Big). \tag{3.37}$$

For $D = 4$, we can obtain traceless ricci tensor (3.17),

$$S_{\mu\nu} = \lambda_\mu\lambda_\nu\mathcal{O}H, \tag{3.38}$$

$$\Box(\lambda_\mu\lambda_\nu H) = \bar{\Box}(\lambda_\mu\lambda_\nu H) = -\lambda_\mu\lambda_\nu\Big(\mathcal{O} + \frac{2}{l^2}\Big). \tag{3.39}$$

Hence, the equation is,

$$\Box(\lambda_\mu\lambda_\nu H) = -\lambda_\mu\lambda_\nu\Bigg(-\Box - 2\xi^\mu\partial_\mu - \frac{1}{2}\xi^\mu\xi_\mu + \frac{6}{l^2}\Bigg). \tag{3.40}$$

Also,

$$\Box S_{\mu\nu} = -\lambda_\mu\lambda_\nu\Bigg(\mathcal{O} + \frac{2}{l^2}\Bigg)\mathcal{O}H. \tag{3.41}$$

To make it more explicit,

$$S_{\mu\nu} = -\Big(\bar{\Box} + \frac{2}{l^2}\Big)(\lambda_\mu\lambda_\nu H) = -\frac{1}{2}\Big(\bar{\Box} + \frac{2}{l^2}\Big)h_{\mu\nu}, \tag{3.42}$$

where $h_{\mu\nu} = 2H\lambda_\mu\lambda_\nu$.

Hence, AdS-plane wave metric $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ with the help of $\partial_\mu \mathcal{O}H = 0$

$$\Box^n S_{\mu\nu} = (-1)^n \lambda_\mu\lambda_\nu \left(\mathcal{O} + \frac{2}{l^2}\right)^n \mathcal{O}H = \bar{\Box}^n S_{\mu\nu}. \tag{3.43}$$

To get the Weyl tensor with the higher order derivative, one can start with

$$\nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} = \frac{1}{2}\Big[\Box S^{\alpha\beta} - \nabla_\mu\nabla^\alpha S^{\mu\beta}\Big], \tag{3.44}$$

for constant curvature spacetime.

Then, if one uses $\nabla_\mu\nabla^\gamma S^{\mu\nu} = \frac{R}{3}S^{\gamma\nu}$ which holds the metric tensor (3.33).

$$\nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} = \frac{1}{2}\Big[\Box S^{\alpha\beta} - \frac{R}{3}S^{\alpha\beta}\Big]. \tag{3.45}$$

Hence, one can reach the equation of the Weyl tensor [9],

$$\nabla_\mu\nabla_\nu\Box^n C^{\mu\alpha\nu\beta} = \Big(\Box + \frac{R}{3}\Big)^n \nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} = \frac{1}{2}\Big(\Box + \frac{R}{3}\Big)^n \Big(\Box - \frac{R}{3}\Big)S^{\alpha\beta}. \tag{3.46}$$

One can change the field equations of the IDG to a simply form [9],

$$\Big(\Lambda + \frac{3}{l^2}\Big)g_{\mu\nu} + \Big[1 + \alpha_c[f_{1,0} + \frac{f_{2,0}}{2}]R + (\bar{\Box} + \frac{2}{l^2})F_2(\bar{\Box}_s) + 2F_3(\bar{\Box}_s - \frac{4}{M_s^2 l^2})(\bar{\Box} + \frac{4}{l^2})\Big]S_{\mu\nu} = 0. \tag{3.47}$$

The trace part of the equation is the cosmological constant,

$$\Lambda = -\frac{3}{l^2}. \tag{3.48}$$

The traceless part gives non local parts,

$$\Big[1 + \alpha_c\big[-\frac{12}{l^2}(2f_{1,0} + \frac{f_{2,0}}{2}) + (\bar{\Box} + \frac{2}{l^2})F_2(\bar{\Box}_s) + 2F_3(\bar{\Box}_s - \frac{4}{M_s^2 l^2})(\bar{\Box} + \frac{4}{l^2})\big]\Big]\Big(\bar{\Box} + \frac{2}{l^2}\Big)\lambda_\mu\lambda_\nu H = 0 \tag{3.49}$$

where

$$R = -\frac{12}{l^2}, \tag{3.50}$$

$$S_{\mu\nu} = \lambda_\mu\lambda_\nu \mathcal{O}H = \lambda_\mu\lambda_\nu(\bar{\Box} + \frac{2}{l^2})H. \tag{3.51}$$

One can choose, the form factors as $F_1 = 0, F_2 = 0, F_3 \neq 0$, the theory reduces to spin-2 excitations as well as no spin-0 mode exist [28].

$$F_1(\Box_s) = F_2(\Box_s) = 0, \tag{3.52}$$

$$F_3(\Box_s) = \frac{1}{2}\frac{e^{-(\Box_s+\frac{8}{l^2M_s^2})}-1}{\Box_s+\frac{8}{l^2M_s^2}}. \tag{3.53}$$

The AdS wave equation (3.49) can be reduced, [9]

$$e^{-\left(\bar{\Box}_s+\frac{2}{M_s^2l^2}\right)}\left(\bar{\Box}+\frac{2}{l^2}\right)\lambda_\mu\lambda_\nu H = 0. \tag{3.54}$$

One can write AdS wave metric by using null coordinates,

$$u = \frac{1}{\sqrt{2}}(x-t), v = \frac{1}{\sqrt{2}}(x+t), \tag{3.55}$$

$$ds^2 = \frac{l^2}{z^2}\left(2dudv + dy^2 + dz^2\right) + 2Hdu^2. \tag{3.56}$$

In the null coordinates,

$$,\xi_\mu = \frac{2}{z}\delta^z_\mu \tag{3.57}$$

$$\mathcal{O} = -\left(\bar{\Box}+\frac{4z}{l^2}\partial_z - \frac{2}{l^2}\right), \tag{3.58}$$

$$\bar{\Box} = \frac{z^2}{l^2}(\partial_z^2+\partial_y^2) - \frac{2z}{l^2}\partial_z - \frac{4z^2}{l^2}\partial_u\partial_v, \tag{3.59}$$

where $\partial^2 = \partial_y^2 + \partial_z^2$.

By using equation (3.39), the field equations (3.54) reduce to new form is,

$$\overbrace{e^{-(\bar{\Box}_s+\frac{2}{M_s^2l^2})}}^{1}\overbrace{\left(\bar{\Box}+\frac{2}{l^2}\right)\lambda_\mu\lambda_\nu H}^{2} = 0, \tag{3.60}$$

$$(1) \longrightarrow e^{-(\bar{\Box}_s+\frac{2}{M_s^2l^2})} = e^{-(\frac{\Box}{M_s^2}+\frac{2}{M_s^2l^2})} = e^{-\frac{1}{M_s^2}\left[\frac{z^2}{l^2}\partial^2-\frac{2z}{l^2}\partial_z-\frac{4z^2}{l^2}\partial_u\partial_v+\frac{2}{l^2}\right]}. \tag{3.61}$$

Hence, the first part of the equation is,

$$e^{-(\bar{\Box}_s+\frac{2}{M_s^2l^2})} = e^{-\frac{1}{M_s^2l^2}\left[z^2\partial^2-2z\partial_z+2\right]}, \tag{3.62}$$

where $\Box_s = \frac{\Box}{M_s^2}$.

$$\begin{aligned}(2) \longrightarrow \left[(\bar{\Box}+\frac{2}{l^2})\lambda_\mu\lambda_\nu H\right] &= \bar{\Box}\lambda_\mu\lambda_\nu H = -\lambda_\mu\lambda_\nu\mathcal{O}H = -\lambda_\mu\lambda_\nu\left[-(\bar{\Box}+\frac{4z}{l^2}\partial_z-\frac{2}{l^2})\right] \\ &= \lambda_\mu\lambda_\nu\left[\frac{z^2}{l^2}\partial^2-\frac{2z}{l^2}\partial_z-\frac{4z^2}{l^2}\partial_u\partial_v\right]+\frac{4z}{l^2}\partial_z\lambda_\mu\lambda_\nu-\frac{2}{l^2}\lambda_\mu\lambda_\nu.\end{aligned} \tag{3.63}$$

Hence, the second part of the equation is,

$$(\bar{\Box} + \frac{2}{l^2})\lambda_\mu\lambda_\nu H = \lambda_\mu\lambda_\nu\frac{z^2}{l^2}\partial^2 + \frac{2z}{l^2}\partial_z\lambda_\mu\lambda_\nu - \frac{2}{l^2}\lambda_\mu\lambda_\nu. \tag{3.64}$$

Therefore, the field equations reduce to

$$e^{-\frac{z^2\partial^2+2z\partial_z-2}{M_s^2 l^2}}[z^2\partial^2 + 2z\partial_Z - 2]H = 0. \tag{3.65}$$

One can think that the eigenvalue problem of the operator can be solved as [27],

$$(z^2\partial^2 + 2z\partial_Z - 2)H_\alpha = -\alpha^2 H_\alpha, \tag{3.66}$$

where H_α are the eigenfunctions and w is the eigenvalues. Hence, the equation reduces to

$$(z^2\partial^2 + 2z\partial_Z - 2)H = 0. \tag{3.67}$$

where $e^{\frac{\alpha^2}{M_s^2 l^2}}\alpha^2 = 0$.

That is, the only AdS wave solutions of the source-free theory are those of the Einstein's general relativity in the AdS background.

3.2.1 Impulsive Waves 3+1 Dimensions

One can search for impulsive gravitational waves that are generated by massles sources in IDG [9] [2]. Since we put a non-zero stress-energy on the right hand side of the equation of motion, one can expect that the resulting solutions will be affected by presence of non-local form factors with infinite derivatives.

3.2.1.1 Massless Point-Like Source

Let us talk about the impulsive AdS wave metric,

$$ds^2 = \frac{l^2}{z^2}(2dudv + dy^2 + dz^2) + 2\delta(u)H(y,z)du^2. \tag{3.68}$$

Consider a massless point particle travelling in the positive x-direction with momentum $p^\mu = E(\delta^\mu_t + \delta^\mu_x)$. The metric for this particle $g_{uu} = 2\delta(u)H(y,z)$, such a particle could be described by a source,

$$T_{uu} = Ez_0^2 l^{-2}\delta(u)\delta(y)\delta(z - z_0). \tag{3.69}$$

[2] In this section, for the details on the impulsive gravitational waves , see [9].

The AdS-wave equation is,

$$e^{-\frac{z^2\partial^2+2z\partial_z-2}{M_s^2 l^2}}(z^2\partial^2+2z\partial_z-2)H(y,z)=-\kappa\delta(y)\delta(z-z_0), \tag{3.70}$$

where the constant $\kappa = 16\pi G E z_0^2$.

In order to solve the equation (3.70), we first take the Fourier transform in coordinate y,

$$e^{-\frac{z^2\partial^2+z^2\partial_y^2+2z\partial_z-2}{M_s^2 l^2}}(z^2\partial^2+z^2\partial_y^2+2z\partial_z-2)H(y,z)=-\kappa\delta(y)\delta(z-z_0). \tag{3.71}$$

Let us define the Fourier transforms as,

$$\tilde{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{\Re} dx f(y)e^{-iky}, \tag{3.72}$$

$$f(y)=\frac{1}{\sqrt{2\pi}}\int_{\Re} dx \tilde{f}(k)e^{iky}. \tag{3.73}$$

Hence, fourier transform, $y \longrightarrow k$

$$H(y,z)=\frac{1}{\sqrt{2\pi}}\int_{\Re} dk \tilde{H}(k,z)e^{iky}, \tag{3.74}$$

$$\delta(y)=\frac{1}{2\pi}\int_{\Re} dk e^{iky}. \tag{3.75}$$

The equation (3.75) turns into,

$$e^{-\frac{z^2\partial^2-z^2k^2+2z\partial_z-2}{M_s^2 l^2}}(z^2\partial^2-z^2k^2+2z\partial_z-2)H(k,z)=-\frac{\kappa}{\sqrt{2\pi}}\delta(z-z_0). \tag{3.76}$$

Using this substitution, $\hat{H}=\frac{V(k,z)}{\sqrt{z}}$, one can rewrite the last equation as,

$$\begin{aligned}(z^2\partial_z^2, 42z\partial_z-k^2z^2-2)\frac{1}{\sqrt{z}}V(k,z)&=\frac{1}{\sqrt{z}}\epsilon(k)V(k,z),\\ \frac{1}{\sqrt{z}}e^{-\frac{\epsilon(k)}{l^2M_s^2}}\epsilon(k)V(k,z)&=-\frac{\kappa}{\sqrt{2\pi}}\delta(z-z_0),\\ e^{-\frac{\epsilon(k)}{l^2M_s^2}}\epsilon(k)V(k,z)&=-\bar{\kappa}\delta(z-z_0),\end{aligned} \tag{3.77}$$

where

$$\bar{\kappa}=\frac{\kappa\sqrt{z_0}}{\sqrt{2\pi}}. \tag{3.78}$$

Let's examine the eigenvalue problem for this operator,

$$\epsilon(k)v^{\alpha} = \alpha v^{\alpha}, \tag{3.79}$$

$$\begin{aligned} v^{\alpha} &= aJ_{\beta}(-ikz) + bY_{\beta}(-ikz), \\ &= \tilde{a}I_{\beta}(kz) + \tilde{b}K_{\beta}(kz), \\ &= \tilde{a}I_{i\bar{\beta}}(kz) + \tilde{b}K_{i\bar{\beta}}(kz), \end{aligned} \tag{3.80}$$

where

$$\beta = \sqrt{\frac{9}{4} + \alpha}, \quad \beta = i\bar{\beta}, \quad \bar{\beta}\ \epsilon\ \mathbf{R}. \tag{3.81}$$

Assuming $k > 0$,

$$\epsilon(k)K_{i\bar{\beta}}(kz) = -(\bar{\beta}^2 + \frac{9}{4})K_{i\bar{\beta}}(kz) \tag{3.82}$$

where

$$\tilde{a} = 0, \quad \tilde{b} = 1. \tag{3.83}$$

where $K_{i\bar{\beta}}$ are Bessel functions of the imaginary order. One can express the right part of the (217) in terms of the eigenfunctions $K_{i\bar{\beta}}$

$$V(\bar{z_0}) = \int_0^{\infty} d\bar{z} \overbrace{\left[\int_0^{\infty} d\bar{\beta}\frac{2}{\pi^2\bar{z}_0}K_{i\bar{\beta}}(\bar{z}_0)\bar{\beta}sinh(\pi\bar{\beta})K_{i\bar{\beta}}(\bar{z})\right]}^{\delta(z-z_0)} f(\bar{z}). \tag{3.84}$$

One can arrange this as [29],

$$\delta(z - z_0) = k\delta(z - z_0) = \int_0^{\infty} d\bar{\beta}\frac{2}{\pi^2\bar{z}_0}K_{i\bar{\beta}}(k\bar{z}_0)\bar{\beta}sinh(\pi\bar{\beta})K_{i\bar{\beta}}(k\bar{z}), \tag{3.85}$$

for arbitrary $k > 0$,

$$\bar{z} = kz, \quad \bar{z_0} = kz_0. \tag{3.86}$$

Now, one can solve equation (3.77),

$$\begin{aligned} V(k, z) &= -\bar{\kappa}\frac{e^{\frac{\epsilon(k)}{M_s^2 l^2}}}{\epsilon(k)}\delta(z - z_0), \\ &= -\bar{\kappa}\int_0^{\infty} d\bar{\beta}\frac{2}{\pi^2 z_0}K_{i\bar{\beta}}(kz_0)\bar{\beta}sinh(\pi\bar{\beta})\frac{e^{\frac{\epsilon(k)}{M_s^2 l^2}}}{\epsilon(k)})K_{i\bar{\beta}}(kz). \end{aligned} \tag{3.87}$$

Substitute $\bar{\kappa}$,

$$V(k, z) = \frac{\sqrt{2}\kappa}{\pi^{\frac{5}{2}}\sqrt{z_0}}\int_0^{\infty} d\bar{\beta}\ \frac{e^{\frac{-(\bar{\beta}^2+\frac{9}{4})}{M_s^2 l^2}}}{\bar{\beta}^2 + \frac{9}{4}}\bar{\beta}sinh(\pi\bar{\beta})\ X\ K_{i\bar{\beta}}(kz_0)K_{i\bar{\beta}}(kz). \tag{3.88}$$

After the arrangement, the solution of (3.76) takes a new form as integral,

For $k > 0$ [9],

$$\hat{H}(k,z) = \frac{16GEz_0^{\frac{3}{2}}}{\pi^2}\frac{1}{\sqrt{z}}\int_{\Re} dk \int_0^{\infty} d\bar{\beta}\frac{e^{\frac{-(\bar{\beta}^2+\frac{9}{4})}{M_s^2 l^2}}}{\bar{\beta}^2+\frac{9}{4}}\bar{\beta}sinh(\pi\bar{\beta})\ X\ K_{i\bar{\beta}}(|k|z_0)K_{i\bar{\beta}}(|k|z)e^{iky} \tag{3.89}$$

For $k < 0$,

$$\hat{H}(k,z) = \hat{H}(-k,z). \tag{3.90}$$

When M_s goes to infinity, $-\frac{(\beta^2+\frac{9}{4})}{M_s^2 l^2}$ goes to zero. Hence the equation reduces to a new form, and by using some special integral rules, one can arrange the equation [30],

$$\int_{\infty}^{0} d\bar{\beta}\ \frac{\bar{\beta}sinh(\pi\bar{\beta})}{\beta^2+\frac{9}{4}}K_{i\bar{\beta}}(|k|z_0)K_{i\bar{\beta}}(|k|z) = \begin{cases} \frac{\pi^2}{2}I_{\frac{3}{2}}(|k|z)K_{\frac{3}{2}}(k|z_0|) & z < z_0, \\ \frac{\pi^2}{2}I_{\frac{3}{2}}(|k|z_0)K_{\frac{3}{2}}(k|z|) & z > z_0. \end{cases} \tag{3.91}$$

One can arrive at the function,

$$H_{GR} = \frac{2GE}{z^2}\left[(y^2+z^2+z_0^2)\ log\ \ (1+\frac{4zz_0}{y^2+(z-z_0)^2}) - 4zz_0\right]. \tag{3.92}$$

This GR solution represents an impulsive gravitational wave that is generated by a massless particle.

The impulsive wave solution of GR diverges at the location of the particle, where it has distributional curvature. In the non-local impulsive wave solution of IDG is regular everywhere because of the improved behavior of the propagator in the UV scale.

3.2.1.2 Massless Linear Source

$T_{uu} = Ez_0 l^{-2}\delta(u)\delta(z-z_0)$ for which one could find an impulsive wave solution. A linear null source which moves in x-direction with $p^\mu = E(\delta_t^\mu + \delta_x^\mu)$ which is momentum and extends to infinity in y-direction. The choice of the source says that the function H could be independent of y. Hence, the field equation takes a simpler form,

$$e^{-\frac{z^2\partial_z^2+2z\partial_z-2}{M_s^2 l^2}}(z^2\partial_z^2+2z\partial_z-2)H(z) = -L_4\delta(z-z_0), \tag{3.93}$$

where $L_4 = 16\pi GEz_0$.

After transforming the equation to the coordinate $w = logz$ and defining $\tilde{H}(w) =$

$H(e^w)$, one can write,

$$\begin{aligned}
\tilde{H}(w) &= -L_4 e^{-w_0} \frac{e^{\frac{(\partial_w^2+\partial_w-2)}{M_s^{l\,2}}}}{(\partial_w^2+\partial_w-2)}\delta(w-w_0), \\
&= L_4 e^{-w_0} \int_{\frac{1}{M_s^2 l^2}}^{\infty} e^{s(\partial_w^2+\partial_w-2)}\delta(w-w_0)ds, \\
&= L_4 e^{-w_0} \int_{\frac{1}{M_s^{l\,2}}}^{\infty} e^{-2s} e^{s\partial_w^2}\delta(w-w_0+s), \\
\tilde{H} &= L_4 e^{-w_0} \int_{\frac{1}{M_s^2 l^2}}^{\infty} ds e^{-2s} \int_{\Re} d\tilde{w} \frac{e^{-\frac{(w-\tilde{w})^2}{4s}}}{\sqrt{4\pi s}}\delta(\tilde{w}-w_0+s).
\end{aligned} \tag{3.94}$$

One can return back to the variable z, one can obtain the solution of (3.93) [9],

$$H(z) = \frac{8\pi GE}{3z^2 z z_0}\Big[z_0^3 erfc\Big(\frac{3}{2M_s l} - \frac{M_s l}{2}log(\frac{z}{z_0})\Big) + z^3 erfc\Big(\frac{3}{2M_s l} + \frac{M_s l}{2}log(\frac{z}{z_0})\Big)\Big], \tag{3.95}$$

which is plotted in figure, where $w \longrightarrow Log(z)$ and $w_0 \longrightarrow Log(z_0)$.

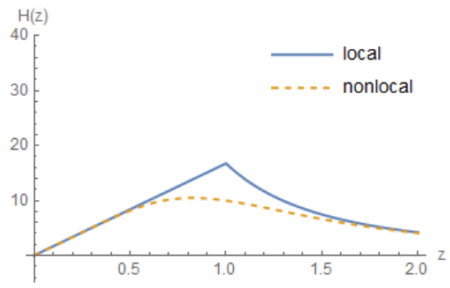

Figure 3.1: The red curve represents the solution of IDG and the blue curve represents the solution of GR (Mathematica).

By calculating with the local limit $M_s \longrightarrow \infty$, one can reach the GR solution,

$$H_{GR} = \frac{8\pi GEz}{3z_0}\Big(1 + \frac{z_0^3}{z^3} - |1 - \frac{z_0^3}{z^3}|\Big). \tag{3.96}$$

The GR solution has a discontinuity at the location of the source $z = z_0$, in contrast to the IDG solution is smooth everywhere.

3.2.2 Impulsive Waves In 2+1 Dimensions

In this subsection, we have followed [9], solutions in 2 + 1 dimensions will be studied. The motivation for this section comes from the fact that the basic parts which are

almost the same, but focus on the crucial differences from the $4D$ case.

Weyl tensor is zero in 2+1 dimensions so the IDG action contains only the form factors. Hence, the traceless part of the source-free field equations in 3-D will be reduced to

$$\left[1+\alpha_c\big[-\frac{12}{l^2}(f_{1,0}+\frac{f_{2,0}}{3})+(\bar{\Box}+\frac{2}{l^2})F_2(\bar{\Box}_s)\big]\right]\times\Big(\bar{\Box}+\frac{2}{l^2}\lambda_\mu\lambda_\nu H\Big)=0 \quad (3.97)$$

One can get this equation from the equation (3.49), where F_3 is zero. One needs to set the form factor $F_2(\Box_s)$ to be in this form to avoid ghost like degrees of freedom. Ghost gives the negative KE and one of the purpose of the IDG is avoid this ghost instabilities.

Now, one can note that the field equation is independent of the form factor $F_1(\Box_s)$. In the next section, solutions in the presence of the non-zero source will be found.

3.2.2.1 Massless Point-Like Source

Consider a point-like particle moving in the positive x-direction with the momentum $p^\mu = E(\delta^\mu_t + \delta^\mu_x)$ with the stress energy tensor $T_{uu} = Ez_0^2 l^{-2}\delta(u)\delta(z-z_0)$. This source together with the impulsive-wave profile $H = \delta(u)H(z)$ leads to a new equation,

$$e^{-\frac{z^2\partial_z^2+3z\partial_z}{M_s^2 l^2}}(z^2\partial_z^2+3z\partial_z)H(z) = -L_3\delta(z-z_0), \quad (3.98)$$

where $L_3 = 16\pi G_3 E z_0^2/C$.

With the help of the $w = logz$, $\tilde{H}(w) = H(e^w)$ and with the Heat-Kernel method,

$$\begin{aligned}\tilde{H}(w) &= -L_3 e^{-w_0}\frac{e^{\frac{(\partial_w^2+2\partial_w)}{M_S^2 l^2}}}{\partial_w^2+2\partial_w}\delta(w-w_0),\\ &= L_3 e^{-w_0}\int_{1/M_s^{l2}}^{\infty} ds\int_{\Re} d\tilde{w}\frac{e^{-\frac{(w-\tilde{w})^2}{4s}}}{\sqrt{4\pi s}}\delta(\tilde{w}-w_0+2s)\end{aligned} \quad (3.99)$$

which is same calculation with the equation (3.94).

The particular solution of equation (3.98) is [9],

$$H(z) = \frac{4\pi G_3 E z_0}{Cz^2}\left[z_0^2 erfc\Big(\frac{1}{M_s l}-\frac{M_s l}{2}log(\frac{z}{z_0})\Big)+z^2 erfc\Big(\frac{1}{M_s l}+\frac{M_s l}{2}log(\frac{z}{z_0})\Big)\right], \quad (3.100)$$

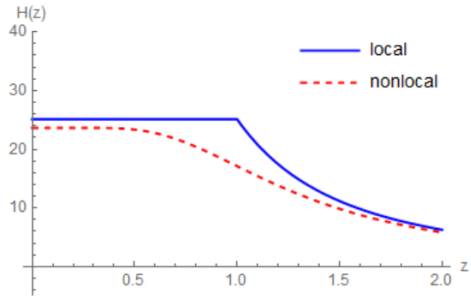

Figure 3.2: The red curve shows the solution of IDG and the blue curve represents the corresponding solution of GR.

which can be plotted in the figure.

By calculating the local limit $M_s \longrightarrow \infty$, one can reach the GR solution,

$$H_{GR} = 4\pi G_3 E z_0 \left(1 + \frac{z_0^2}{z^2} - |1 - \frac{z_0^2}{z^2}|\right) \tag{3.101}$$

3-D GR solution is stable but has a discontinuity at $z = z_0$. This discontinuity problem is solved by infinite derivatives. The IDG impulsive wave solution is smooth everywhere. The full solution approaches the general relativity solution at the conformal infinity $(z = 0)$. The metric of the GR solution is just the AdS metric.

CHAPTER 4

CONCLUSIONS AND LAST REMARKS

In this thesis, we have firstly studied pp-wave space-times in ghost-free infinite derivative gravity and shown that these space-times are exact solutions of the theory. The pp-wave metrics also solve linearized field equations of the IDG for the pp-wave space-time. By using the metric in the Kerr-Schild form gives an important simplification on the field equations. We have demonstrated that, as expected, sourceless theory does not bring any pp-wave solutions other than that of GR since non-local interactions do not affect source-free linear field equations. To discuss the effects of non-local interactions, we have considered null source coupled field equations and found exact gravitational shock wave solution of the theory.

Secondly, we have studied the pp-wave solutions of the IDG as in the pp-wave case, the exact AdS-plane wave solutions of sourceless theory are also solutions of GR. In the presence of a source, we constructed impulsive waves created by massless sources in 2+1 and 3+1 dimensions. The non-locality described by form factors with higher derivatives which plays an important role in the non-zero source. The solutions which have obtained of the IDG are regular everywhere due to non-local interactions. The solutions which we have found get modified because of the non local impacts in the UV part, but not in the IR part.

The field equations of IDG were examined. Symmetric tensors were given in the chapter 2 which containing the double sums and they are too hard to solve explicitly. We determined the full equations of motion and we can see that if one keeps the terms linear in curvature, can find out the linearized limit in flat-space without using any new degrees of freedom which is expected.

We observed that even though gravitational shock-wave solution comes with a source which has Dirac delta type singularity, the solution is regular at the location of source

because of the gravitational non local interactions. Then, we have found a non-singular gravitational shock-wave solution at the non-linear level.

The solutions given in this thesis are crucial solutions. Few exact solutions exist in the quadratic gravity theories and gravitational waves have lots of unexplored possibilities. Also, the quadratic theories are rare in the literature, so these are very interesting.

References

[1]L. Modesto, "Super-renormalizable quantum gravity", *Phys. Rev. D*, vol. 86, p. 044 005, 4 Aug. 2012. DOI: 10.1103/PhysRevD.86.044005. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevD.86.044005.

[2]T. Biswas, A. Mazumdar, and W. Siegel, "Bouncing universes in string-inspired gravity", *Journal of Cosmology and Astroparticle Physics*, vol. 2006, no. 03, pp. 009–009, Mar. 2006, ISSN: 1475-7516. DOI: 10.1088/1475-7516/2006/03/009. [Online]. Available: http://dx.doi.org/10.1088/1475-7516/2006/03/009.

[3]T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, "Towards singularity- and ghost-free theories of gravity", *Physical Review Letters*, vol. 108, no. 3, Jan. 2012, ISSN: 1079-7114. DOI: 10.1103/physrevlett.108.031101. [Online]. Available: http://dx.doi.org/10.1103/PhysRevLett.108.031101.

[4]T. Biswas, A. Conroy, A. Koshelev, and A. Mazumdar, "Generalized ghost-free quadratic curvature gravity", English, *Classical and Quantum Gravity*, vol. 31, Nov. 2014, [Erratum: Class. Quant. Grav.31,159501(2014)], ISSN: 0264-9381. DOI: 10.1088/0264-9381/31/1/015022.

[5]A. S. Koshelev, J. Marto, and A. Mazumdar, "Schwarzschild 1/r singularity is not permissible in ghost-free quadratic-curvature infinite-derivative gravity", *Physical Review D*, vol. 98, no. 6, Sep. 2018, ISSN: 2470-0029. DOI: 10.1103/physrevd.98.064023. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.98.064023.

[6]E. Kilicarslan, "Weak field limit of infinite derivative gravity", *Physical Review D*, vol. 98, no. 6, Sep. 2018, ISSN: 2470-0029. DOI: 10.1103/physrevd.98.064048. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.98.064048.

[7]T. Biswas, T. Koivisto, and A. Mazumdar, "Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity", *Journal of Cosmology and Astroparticle Physics*, vol. 2010, no. 11, pp. 008–008, Nov. 2010, ISSN: 1475-7516. DOI: 10.1088/1475-7516/2010/11/008. [Online].

Available: http://dx.doi.org/10.1088/1475-7516/2010/11/008.

[8]E. Kilicarslan, "Pp -waves as exact solutions to ghost-free infinite derivative gravity", *Physical Review D*, vol. 99, no. 12, Jun. 2019, ISSN: 2470-0029. DOI: 10.1103/physrevd.99.124048. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.99.124048.

[9]S. Dengiz, E. Kilicarslan, I. Kolář, and A. Mazumdar, "Impulsive waves in ghost-free infinite derivative gravity in anti–de sitter spacetime", *Physical Review D*, vol. 102, no. 4, Aug. 2020, ISSN: 2470-0029. DOI: 10.1103/physrevd.102.044016. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.102.044016.

[10]I. Kolář, T. Málek, and A. Mazumdar, "Exact solutions of nonlocal gravity in a class of almost universal spacetimes", *Physical Review D*, vol. 103, no. 12, Jun. 2021, ISSN: 2470-0029. DOI: 10.1103/physrevd.103.124067. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.103.124067.

[11]I. Kolář, T. Málek, S. Dengiz, and E. Kilicarslan, *Exact gyratons in higher and infinite derivative gravity*, 2021. arXiv: 2107.11884 [gr-qc].

[12]K. S. Kumar, S. Maheshwari, A. Mazumdar, and J. Peng, "Stable, nonsingular bouncing universe with only a scalar mode", *Physical Review D*, vol. 102, no. 2, Jul. 2020, ISSN: 2470-0029. DOI: 10.1103/physrevd.102.024080. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.102.024080.

[13]L. Buoninfante, A. S. Koshelev, G. Lambiase, and A. Mazumdar, "Classical properties of non-local, ghost- and singularity-free gravity", *Journal of Cosmology and Astroparticle Physics*, vol. 2018, no. 09, pp. 034–034, Sep. 2018, ISSN: 1475-7516. DOI: 10.1088/1475-7516/2018/09/034. [Online]. Available: http://dx.doi.org/10.1088/1475-7516/2018/09/034.

[14]J. W. Moffat, "Ultraviolet complete quantum gravity", *The European Physical Journal Plus*, vol. 126, no. 4, Apr. 2011, ISSN: 2190-5444. DOI: 10.1140/epjp/i2011-11043-7. [Online]. Available: http://dx.doi.org/10.1140/epjp/i2011-11043-7.

[15]J. Edholm, *Infinite derivative gravity: A finite number of predictions*, 2019. arXiv: 1904.10248 [gr-qc].

[16]K. Muneyuki and N. Ohta, "Renormalization of higher derivative quantum gravity coupled to a scalar with shift symmetry", *Physics Letters B*, vol. 725, no. 4-5, pp. 495–499, Oct. 2013, ISSN: 0370-2693. DOI: 10.1016/j.physletb.2013.07.054. [Online]. Available: http://dx.doi.org/10.1016/j.physletb.2013.07.054.

[17]J. Edholm, "Gravitational radiation in infinite derivative gravity and connections to effective quantum gravity", *Physical Review D*, vol. 98, no. 4, Aug. 2018, ISSN: 2470-0029. DOI: 10.1103/physrevd.98.044049. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.98.044049.

[18]A. Conroy, A. Mazumdar, and A. Teimouri, "Wald entropy for ghost-free, infinite derivative theories of gravity", *Physical Review Letters*, vol. 114, no. 20, May 2015, ISSN: 1079-7114. DOI: 10.1103/physrevlett.114.201101. [Online]. Available: http://dx.doi.org/10.1103/PhysRevLett.114.201101.

[19]M. Gürses, T. Sisman, and B. Tekin, "Ads-plane wave and pp-wave solutions of generic gravity theories", *Physical Review D*, vol. 90, p. 124 005, Jul. 2014. DOI: 10.1103/PhysRevD.90.124005.

[20]M. Blau, M. Borunda, M. O'Loughlin, and G. Papadopoulos, "Penrose limits and spacetime singularities", *Classical and Quantum Gravity*, vol. 21, no. 7, pp. L43–L49, Mar. 2004, ISSN: 1361-6382. DOI: 10.1088/0264-9381/21/7/l02. [Online]. Available: http://dx.doi.org/10.1088/0264-9381/21/7/L02.

[21]J. Edelstein, G. Giribet, C. Gomez, E. Kilicarslan, M. Leoni, and B. Tekin, "Causality in 3d massive gravity theories", English, *Physical Review D*, vol. 95, no. 10, May 2017, ISSN: 1550-7998. DOI: 10.1103/PhysRevD.95.104016.

[22]S. Deser and B. Tekin, "Energy in generic higher curvature gravity theories", *Physical Review D*, vol. 67, no. 8, Apr. 2003, ISSN: 1089-4918. DOI: 10.1103/physrevd.67.084009. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.67.084009.

[23]M. Gürses, T. Ç. Şişman, and B. Tekin, "New exact solutions of quadratic curvature gravity", *Physical Review D*, vol. 86, no. 2, Jul. 2012, ISSN: 1550-2368. DOI: 10.1103/physrevd.86.024009. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.86.024009.

[24]M. Gürses, S. Hervik, T. Sisman, and B. Tekin, "Anti–de sitter–wave solutions of higher derivative theories", *Physical Review Letters*, vol. 111, Sep. 2013. DOI: 10.1103/PhysRevLett.111.101101.

[25]I. Gullu, *Massive higher derivative gravity theories*, 2012. arXiv: 1201.6504 [gr-qc].

[26]İ. Güllü, M. Gürses, T. Ç. Şişman, and B. Tekin, "Ads waves as exact solutions to quadratic gravity", *Physical Review D*, vol. 83, no. 8, Apr. 2011, ISSN: 1550-2368. DOI: 10.1103/physrevd.83.084015. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.83.084015.

[27]N. Barnaby and N. Kamran, "Dynamics with infinitely many derivatives: Variable coefficient equations", *Journal of High Energy Physics*, vol. 2008, no. 12, pp. 022–022, Dec. 2008, ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/12/022. [Online]. Available: http://dx.doi.org/10.1088/1126-6708/2008/12/022.

[28]T. Biswas, A. S. Koshelev, and A. Mazumdar, "Consistent higher derivative gravitational theories with stable de sitter and anti–de sitter backgrounds", *Physical Review D*, vol. 95, no. 4, Feb. 2017, ISSN: 2470-0029. DOI: 10.1103/physrevd.95.043533. [Online]. Available: http://dx.doi.org/10.1103/PhysRevD.95.043533.

[29]S. MURASHIMA and T. KIYONO, "Modified bessel functions of purely", no. 15, pp. 91–131, 1973.

[30]I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Seventh. Elsevier/Academic Press, Amsterdam, 2007, pp. xlviii+1171, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX), ISBN: 978-0-12-373637-6; 0-12-373637-4.

Appendix A

USEFUL FORMULAS AND BIANCHI IDENTITIES

A.1 Curvature

The christoffel symbol formula is,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\tau}(\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu}) \tag{A.1}$$

The Riemann tensor and its properties are,

$$R^{\lambda}{}_{\mu\sigma\alpha} = \partial_\sigma \Gamma^{\lambda}_{\mu\alpha} - \partial_\alpha \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\rho}_{\alpha\mu} - \Gamma^{\lambda}_{\alpha\rho}\Gamma^{\rho}_{\sigma\mu} \tag{A.2}$$

$$R_{\rho\sigma\mu\alpha} = g_{\rho\lambda}(\partial_\mu \Gamma^{\lambda}_{\alpha\sigma} - \partial_\alpha \Gamma^{\lambda}_{\mu\sigma}) \tag{A.3}$$

$$R_{\mu\alpha\lambda\sigma} = -R_{\alpha\mu\lambda\sigma} = R_{\mu\alpha\sigma\lambda} = R_{\lambda\sigma\mu\alpha} \tag{A.4}$$

The Ricci tensor is symettric,

$$R_{\mu\nu} = R_{\nu\mu} \tag{A.5}$$

The Ricci scalar is,

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial^\mu \Gamma^{\alpha}_{\mu\alpha} + g^{\mu\nu}\Gamma^{\alpha}_{\alpha\rho}\Gamma^{\rho}_{\nu\mu} - g^{\mu\nu}\Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\alpha_\mu} \tag{A.6}$$

A.2 Bianchi Identities and Riemann Tensor Properties

The Bianchi Identity equation is the fundamental equations to find the Einstein equation.

In general relativity and tensor calculus, the contracted Bianchi identities,

$$\nabla^\mu G_{\mu\nu} = 0, \tag{A.7}$$

where $G_{\mu\nu}$ is the Einstein tensor such that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{A.8}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. One can write this as,

$$\nabla_\mu R^\mu{}_\nu = \frac{1}{2} \nabla_\nu R. \tag{A.9}$$

The usual Bianchi identity is

$$\nabla_\mu R_{\nu\lambda\rho}{}^\delta + \nabla_\nu R_{\lambda\mu\rho}{}^\delta + \nabla_\lambda R_{\mu\nu\rho}{}^\delta = 0 \tag{A.10}$$

where $R^\delta{}_{\nu\lambda\rho}$ is the Riemann tensor.

Remember the following properties of the Riemann tensor

$$R_{\mu\nu\lambda}{}^\rho = -R_{\nu\mu\lambda}{}^\rho, \quad R_{\mu\nu\lambda}{}^\rho = -R_{\mu\nu}{}^\rho{}_\lambda, \quad R_{\mu\nu\lambda}{}^\rho = R_\lambda{}^\rho{}_{\mu\nu} \tag{A.11}$$

A.3 Bianchi Identities for the Weyl Tensor

Once contracted Bianchi identity is

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu R_{\alpha\beta} - \nabla_\alpha R_{\mu\beta}, \tag{A.12}$$

for constant R, gives

$$\nabla^\nu R_{\mu\alpha\nu\beta} = \nabla_\mu S_{\alpha\beta} - \nabla_\nu S_{\mu\beta}. \tag{A.13}$$

Contracting one more time, one has

$$\nabla^\mu S_{\mu\nu} = 0 \quad for \quad R : const. \tag{A.14}$$

and we can get

$$\nabla^\mu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2} (\nabla_\nu S_{\beta\alpha} - \nabla_\beta S_{\nu\alpha}) \tag{A.15}$$

contracted once, for constant curvature spacetimes.

Then, $\nabla^\mu \nabla^\nu C_{\mu\alpha\nu\beta}$ will be

$$\nabla^\mu \nabla^\nu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2} (\Box S_{\alpha\beta} - \nabla^\mu \nabla_\alpha S_{\mu\beta}) \tag{A.16}$$

Then, using

$$\nabla^\mu \nabla_\sigma S_{\mu\nu} = \frac{R}{D-1} S_{\sigma\nu}. \tag{A.17}$$

We can get

$$\nabla^\mu \nabla^\nu C_{\mu\alpha\nu\beta} = \frac{D-3}{D-2} (\Box S_{\alpha\beta} - \frac{R}{D-1} S_{\alpha\beta}) \tag{A.18}$$

A.4 The Specific Calculation

One can obtain the equation (3.46), I will show how to get this equation step by step,

$$\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} = \nabla_\mu[\nabla_\nu,\nabla_\sigma]\nabla^\sigma C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta}$$
$$= \nabla_\mu\Bigg[R_{\nu\sigma}{}^{\sigma}{}_{\lambda}\nabla^\lambda C^{\mu\alpha\nu\beta} + R_{\nu\sigma}{}^{\mu}{}_{\lambda}\nabla^\sigma C^{\lambda\alpha\nu\beta} + R_{\nu\sigma}{}^{\alpha}{}_{\lambda}\nabla^\sigma C^{\mu\lambda\nu\beta} + R_{\nu\sigma}{}^{\nu}{}_{\lambda}\nabla^\sigma C^{\mu\alpha\lambda\beta}$$
$$R_{\nu\sigma}{}^{\beta}{}_{\lambda}\nabla^\sigma C^{\mu\alpha\nu\lambda}\Bigg] + \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta}$$
$$= \nabla_\mu\Bigg[-R_{\nu\lambda}\nabla^\lambda C^{\mu\alpha\nu\beta} + \nabla_\lambda C^{\lambda\alpha\mu\beta} + \nabla_\lambda C^{\mu\lambda\alpha\beta} + R_{\sigma\lambda}C^{\mu\alpha\lambda\beta} + \nabla_\lambda C^{\mu\alpha\beta\lambda}\Bigg]$$
$$+\nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta}$$
$$= \nabla_\mu\Bigg[\nabla_\lambda(C^{\mu\beta\lambda\alpha} + C^{\mu\lambda\alpha\beta} + C^{\mu\alpha\beta\lambda})\Bigg] + \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta}$$
$$\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} = \nabla_\mu\nabla_\nu\nabla_\sigma\nabla^\sigma C^{\mu\alpha\nu\beta} \tag{A.19}$$

where

$$\nabla_\mu\nabla_\lambda(C^{\mu[\beta\lambda\alpha]}) = 0 \tag{A.20}$$

The covariant derivative of the Weyl tensor comes as zero.

$$R_{\nu\alpha}\nabla^\lambda C^{\mu\alpha\nu\beta} = \rho\lambda_\nu\lambda_\lambda C^{\mu\alpha\nu\beta} = -\rho\lambda_\lambda\lambda_\nu\nabla^\lambda C^{\mu\alpha\nu\beta} = 0 \tag{A.21}$$

$$\nabla_\mu[\nabla_\nu,\nabla_\sigma]\nabla^\sigma C^{\mu\alpha\nu\beta} = \nabla_\mu\nabla_\nu\nabla_\sigma C^{\mu\alpha\nu\beta} - \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta} \tag{A.22}$$

If we try to solve the last equation, we are going to start with,

$$\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} = \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma C^{\mu\alpha\nu\beta}$$
$$= \nabla_\mu\Box\nabla_\nu C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla^\sigma[\nabla_\nu,\nabla_\sigma]C^{\mu\alpha\nu\beta}$$
$$= \nabla_\mu\Box\nabla_\nu C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla^\sigma[R_{\nu\sigma}{}^{\mu}{}_{\lambda}C^{\lambda\alpha\nu\beta} + R_{\nu\sigma}{}^{\alpha}{}_{\lambda}C^{\mu\lambda\nu\beta} + R_{\nu\sigma}{}^{\nu}{}_{\lambda}C^{\mu\alpha\lambda\beta} + R_{\nu\sigma}{}^{\beta}{}_{\lambda}C^{\mu\alpha\nu\lambda}]$$
$$= \nabla_\mu\Box\nabla_\nu C^{\mu\alpha\lambda\beta} + \nabla_\mu\nabla^\sigma\Bigg[\frac{R}{12}(C_\sigma{}^{\alpha\mu\beta} + C^\mu{}_\sigma{}^{\alpha\beta} + C^{\mu\alpha\beta}{}_\sigma) + R_{\sigma\lambda}C^{\mu\alpha\lambda\beta}\Bigg]$$
$$= \nabla_\mu\Box\nabla_\nu C^{\mu\alpha\lambda\beta} + \nabla_\mu\nabla^\sigma\Bigg[\frac{R}{12}(C_\sigma{}^{\alpha\mu\beta} + C_\sigma{}^{\mu\beta\alpha} + C_\sigma{}^{\beta\alpha\mu}) + (-\frac{3}{l^2}g_{\sigma\lambda})C^{\mu\alpha\lambda\beta}\Bigg] \tag{A.23}$$

where

$$R_{\sigma\lambda} = \frac{3}{l^2} g_{\sigma\lambda} + \lambda_\sigma \lambda_\lambda \mathcal{O} \tag{A.24}$$

$$\begin{aligned}
&= \nabla_\mu \Box \nabla_\nu C^{\mu\alpha\lambda\beta} + \nabla_\mu \nabla^\sigma \left[\frac{R}{12} \overbrace{(C_\sigma{}^{[\alpha\mu\beta]})}^{=0} - \frac{3}{l^2} C^{\mu\alpha}{}_\sigma{}^\beta \right] \\
&= \nabla_\mu \Box \nabla_\nu C^{\mu\alpha\lambda\beta} - \frac{3}{l^2} [\nabla_\mu \nabla_\sigma C^{\mu\alpha\sigma\beta}] \\
&= \nabla_\mu \Box \nabla_\nu C^{\mu\alpha\lambda\beta} - \frac{3}{2l^2} [\Box S^{\alpha\beta} - \frac{R}{3} S^{\alpha\beta}]
\end{aligned} \tag{A.25}$$

where

$$\begin{aligned}
\nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} &= \frac{1}{2} [\Box S^{\alpha\beta} - \frac{R}{3} S^{\alpha\beta}] \\
&= \nabla_\mu \Box \nabla_\nu C^{\mu\alpha\lambda\beta} - \frac{3}{2l^2} [\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}]
\end{aligned} \tag{A.26}$$

If we look at for the 4-D,

$$R = -\frac{12}{l^2} \tag{A.27}$$

$$\nabla_\mu \nabla_\nu \Box C^{\mu\alpha\nu\beta} = \nabla_\mu \Box \nabla_\nu C^{\mu\alpha\nu\beta} - \frac{3}{2l^2} [\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}] \tag{A.28}$$

Let's define a new value,

$$A^{\alpha\beta} = -\frac{3}{2l^2} [\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}] \tag{A.29}$$

Then,

$$\begin{aligned}
\nabla_\mu \nabla_\nu \Box C^{\mu\alpha\nu\beta} &= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + [\nabla_\beta, \nabla_\sigma] \nabla^\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} + [R_{\mu\sigma}{}^\sigma{}_\lambda \nabla^\lambda \nabla_\nu C^{\mu\alpha\nu\beta} + R_{\mu\sigma\nu}{}^\lambda \nabla^\sigma \nabla_\lambda C^{\mu\alpha\nu\beta} \\
&\quad + R_{\mu\sigma}{}^\mu{}_\lambda \nabla^\sigma \nabla_\nu C^{\lambda\alpha\nu\beta} + R_{\mu\sigma}{}^\beta{}_\lambda \nabla^\sigma \nabla_\nu C^{\mu\lambda\nu\beta} + R_{\mu\sigma}{}^\nu{}_\lambda \nabla^\sigma \nabla_\nu C^{\mu\alpha\lambda\beta}] \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} + [-R_{\mu\lambda} \nabla^\lambda \nabla_\nu C^{\mu\alpha\nu\beta} + R_{\sigma\lambda} \nabla^\sigma \nabla_\nu C^{\lambda\alpha\nu\beta} \\
&\quad + \frac{R}{12} (-\nabla_\nu \nabla_\lambda C^{\lambda\alpha\nu\beta} + \nabla_\lambda \nabla_\nu C^{\alpha\lambda\nu\beta} + \nabla_\lambda \nabla_\nu C^{\nu\alpha\lambda\beta} + \nabla_\lambda \nabla_\nu C^{\beta\alpha\nu\lambda})] \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} + \frac{R}{12} \nabla_\nu \nabla_\lambda [-C^{\lambda\alpha\nu\beta} + C^{\alpha\lambda\nu\beta} + C^{\nu\alpha\lambda\beta} + C^{\beta\alpha\nu\lambda}] \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} + \frac{R}{12} \nabla_\nu \nabla_\lambda [-C^{\lambda\alpha\nu\beta} + \overbrace{C^{\nu[\beta\alpha\lambda]}}^{=0}] \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + A^{\alpha\beta} + \frac{1}{l^2} \nabla_\nu \nabla_\lambda C^{\nu\beta\lambda\alpha} \\
&= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} - \frac{3}{2l^2} [\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}] + \frac{1}{2l^2} [\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}]
\end{aligned} \tag{A.30}$$

Hence, the equation is,

$$\nabla_\mu \nabla_\nu \Box C^{\mu\alpha\nu\beta} = \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} \tag{A.31}$$

where

$$B^{\alpha\beta} = -\frac{1}{l^2}(\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta}) \tag{A.32}$$

When we substitute $B^{\alpha\beta}$ instead of the (A.32), calculations continue in the same way,

$$\begin{aligned}
\nabla_\mu \nabla_\nu \Box C^{\mu\alpha\nu\beta} &= \nabla^\sigma \nabla_\mu \nabla_\sigma \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + \nabla^\sigma [\nabla_\mu, \nabla_\sigma] \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} + \nabla^\sigma [R_{\mu\sigma\nu}{}^\lambda \nabla_\lambda C^{\mu\alpha\nu\beta} + R_{\mu\sigma}{}^\mu{}_\lambda \nabla_\nu C^{\lambda\alpha\nu\beta} + R_{\mu\sigma}{}^\alpha{}_\lambda \nabla_\nu C^{\mu\lambda\nu\beta} \\
&\quad + R_{\mu\sigma}{}^\nu{}_\lambda \nabla_\nu C^{\mu\alpha\lambda\beta} + R_{\mu\sigma}{}^\beta{}_\lambda \nabla_\nu C^{\mu\alpha\nu\lambda}] \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} + \nabla^\sigma \Bigg[\frac{R}{12}(-\nabla_\mu C^{\mu\alpha}{}_\sigma{}^\beta + \nabla_\nu C^\alpha{}_\sigma{}^{\nu\beta} + \nabla_\mu C^{\mu\alpha}{}_\sigma{}^\beta + \nabla_\nu C^{\beta\alpha\nu}{}_\sigma) \\
&\quad + R_{\sigma\lambda} \nabla_\nu C^{\lambda\alpha\nu\beta} \Bigg] \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} + \nabla_\sigma \left[\frac{R}{12}(\nabla_\mu [-C^{\mu\alpha\sigma\beta} + C^{\mu\beta\alpha\sigma} + C^{\mu\alpha\sigma\beta} + C^{\mu\sigma\beta\alpha}]) - \frac{3}{l^2} \nabla_\nu C^{\sigma\alpha\nu\beta} \right] \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} + \nabla_\sigma \left[-\frac{1}{l^2} \nabla_\mu (-C^{\mu\alpha\sigma\beta} + C^{\mu[\beta\alpha\sigma]}) - \frac{3}{l^2} \nabla_\mu C^{\mu\beta\sigma\alpha} \right] \\
&= \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} + B^{\alpha\beta} + -\frac{2}{l^2} [\nabla_\sigma \nabla_\mu C^{\mu\alpha\sigma\beta}]
\end{aligned} \tag{A.33}$$

Hence, the equation turns into the

$$\nabla_\mu \nabla_\nu \Box C^{\mu\alpha\nu\beta} = \Box \nabla_\mu \nabla_\nu C^{\mu\alpha\nu\beta} - \frac{2}{l^2} \Big(\Box S^{\alpha\beta} + \frac{4}{l^2} S^{\alpha\beta} \Big) \tag{A.34}$$

$$
\begin{aligned}
\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} &= \nabla_\mu\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} \\
&= \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} + \nabla_\mu[\nabla_\nu,\nabla_\sigma]\nabla^\sigma\Box C^{\mu\alpha\nu\beta} \\
&= \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\Big[R_{\nu\sigma}{}^{\sigma}{}_{\lambda}\nabla^\lambda\Box C^{\mu\alpha\nu\beta} + R_{\nu\sigma}{}^{\mu}{}_{\lambda}\nabla^\sigma\Box C^{\lambda\alpha\nu\beta} \\
&\quad + R_{\nu\sigma}{}^{\alpha}{}_{\lambda}\nabla^\sigma\Box C^{\mu\lambda\nu\beta} + R_{\nu\sigma}{}^{\nu}{}_{\lambda}\nabla^\sigma\Box C^{\mu\alpha\lambda\beta} + R_{\nu\sigma}{}^{\beta}{}_{\lambda}\nabla^\sigma\Box C^{\mu\alpha\nu\lambda}\Big] \\
&= \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\Big[\overbrace{-R_{\nu\lambda}\nabla^\lambda\Box C^{\mu\alpha\nu\beta}}^{=0} + \overbrace{R_{\sigma\lambda}\nabla^\sigma\Box C^{\mu\alpha\lambda\beta}}^{=0} + \nabla_\mu\Box C^{\lambda\alpha\mu\beta} \\
&\quad + \nabla_\lambda\Box C^{\mu\lambda\alpha\beta} + \nabla_\lambda\Box C^{\mu\alpha\beta\lambda}\Big] \\
&= \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla_\lambda\Box\overbrace{\Big[C^{\mu\beta\lambda\alpha} + C^{\mu\alpha\beta\lambda} + C^{\mu\lambda\alpha\beta}\Big]}^{C^{\mu[\beta\lambda\alpha]=0}}
\end{aligned}
\tag{A.35}
$$

The equations continue with the next calculation part, we are going to give new definition as $C^{\alpha\beta}$.

Therefore, the equation is

$$
\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} = \nabla_\mu\nabla_\sigma\nabla_\nu\nabla^\sigma\Box C^{\mu\alpha\nu\beta} \tag{A.36}
$$

$$
\begin{aligned}
\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} &= \nabla_\mu\nabla^\sigma\nabla_\nu\nabla_\sigma\Box C^{\mu\alpha\nu\beta} \\
&= \nabla_\mu\nabla^\sigma\nabla_\sigma\nabla_\nu\Box C^{\mu\alpha\beta} + \nabla_\mu\nabla^\sigma[\nabla_\nu,\nabla_\sigma]\Box C^{\mu\alpha\nu\beta} \\
&= \nabla_\mu\Box\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla^\sigma\Big[R_{\nu\sigma}{}^{\mu}{}_{\lambda}\Box C^{\lambda\alpha\nu\beta} + R_{\nu\sigma}{}^{\alpha}{}_{\lambda}\Box C^{\mu\lambda\nu\beta} + R_{\nu\sigma}{}^{\nu}{}_{\lambda}\Box C^{\mu\alpha\lambda\beta} + R_{\nu\sigma}{}^{\beta}{}_{\lambda}\Box C^{\mu\alpha\nu\lambda}\Big] \\
&= \nabla_\mu\Box\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla^\sigma\Big[\frac{R}{12}(\Box C_\sigma{}^{\alpha\mu\beta} + \Box C^\mu{}_\sigma{}^{\alpha\beta} + \Box C^{\mu\alpha\beta}{}_\sigma) + R_{\sigma\lambda}\Box C^{\mu\alpha\lambda\beta}\Big] \\
&= \nabla_\mu\Box\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla_\sigma\Big[\frac{R}{12}\Box(C^{\mu\beta\sigma\alpha} + C^{\mu\sigma\alpha\beta} + C^{\mu\alpha\beta\sigma}) - \frac{3}{l^2}\Box C^{\mu\alpha\sigma\beta}\Big] \\
&= \nabla_\mu\Box\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \nabla_\mu\nabla_\sigma\Big[\frac{R}{12}(\overbrace{C^{\mu[\beta\sigma\alpha]}}^{=0}) - \frac{3}{l^2}\Box C^{\mu\alpha\sigma\beta}\Big]
\end{aligned}
\tag{A.37}
$$

One can examined, the other denominations, and try to simplify these parts,

$$\nabla_\mu \nabla_\nu \Box^2 C^{\mu\alpha\nu\beta} = \nabla_\mu \Box \nabla_\nu \Box C^{\mu\alpha\nu\beta} - \frac{3}{l^2} \nabla_\mu \nabla_\sigma \Box C^{\mu\alpha\sigma\beta} \tag{A.38}$$

$$\nabla_\mu \nabla_\nu \Box^2 C^{\mu\alpha\nu\beta} = \nabla_\mu \Box \nabla_\nu \Box C^{\mu\alpha\nu\beta} + C^{\alpha\beta} \tag{A.39}$$

where

$$C^{\alpha\beta} = -\frac{3}{l^2} \nabla_\mu \nabla_\sigma \Box C^{\mu\alpha\sigma\beta} \tag{A.40}$$

If we try to solve the last part,

$$\begin{aligned}
\nabla_\mu \nabla_\nu \Box^2 C^{\mu\alpha\nu\beta} &= \nabla_\sigma \nabla_\mu \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} + C^{\alpha\beta} + [\nabla_\mu, \nabla_\sigma] \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} \\
&= \nabla_\sigma \nabla_\mu \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} + C^{\alpha\beta} + \Big[R_{\mu\sigma}{}^{\sigma}{}_{\lambda} \nabla^\lambda \nabla_\nu \Box C^{\mu\alpha\nu\beta} + R_{\mu\sigma\nu}{}^{\lambda} \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} \\
&R_{\mu\sigma}{}^{\mu}{}_{\lambda} \nabla^\sigma \nabla_\nu \Box C^{\lambda\alpha\nu\beta} + R_{\mu\sigma}{}^{\alpha}{}_{\lambda} \nabla^\sigma \nabla_\nu \Box C^{\mu\lambda\nu\beta} + R_{\mu\sigma}{}^{\nu}{}_{\lambda} \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\lambda\beta} + R_{\mu\sigma}{}^{\beta}{}_{\lambda} \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\lambda} \Big] \\
&= \nabla_\sigma \nabla_\mu \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} + C^{\alpha\beta} + \Big[- R_{\mu\lambda} \nabla^\lambda \nabla_\nu \Box C^{\mu\alpha\nu\beta} + R_{\sigma\lambda} \nabla^\sigma \nabla_\nu \Box C^{\lambda\alpha\nu\beta} \\
&+ \frac{R}{12} (-\nabla_\nu \nabla_\lambda \Box C^{\lambda\alpha\nu\beta} + \nabla_\lambda \nabla_\nu \Box C^{\alpha\lambda\nu\beta} + \nabla_\lambda \nabla_\nu \Box C^{\nu\alpha\lambda\beta} + \nabla_\lambda \nabla_\nu \Box C^{\beta\alpha\nu\lambda}) \Big] \\
&= \nabla_\sigma \nabla_\mu \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} + C^{\alpha\beta} + \frac{R}{12} \Big[- \nabla_\nu \nabla_\lambda \Box C^{\lambda\alpha\nu\beta} + \nabla_\lambda \nabla_\nu \Box (C^{\nu[\beta\alpha\lambda]}) \Big] \\
&= \nabla_\sigma \nabla_\mu \nabla^\sigma \nabla_\nu \Box C^{\mu\alpha\nu\beta} - \frac{3}{l^2} \nabla_\mu \nabla_\sigma \Box C^{\mu\alpha\sigma\beta} + \frac{1}{l^2} \nabla_\nu \nabla_\lambda \Box C^{\nu\beta\lambda\alpha} \\
\nabla_\mu \nabla_\nu \Box^2 C^{\mu\alpha\nu\beta} &= \nabla_\sigma \nabla_\nu \nabla^\sigma \nabla_\nu C^{\mu\alpha\nu\beta} - \frac{2}{l^2} \nabla_\mu \nabla_\sigma \Box C^{\mu\alpha\sigma\beta}
\end{aligned} \tag{A.41}$$

Let's define a new term,

$$D^{\alpha\beta} = -\frac{2}{l^2} \nabla_\mu \nabla_\sigma \Box C^{\mu\alpha\sigma\beta} \tag{A.42}$$

$$
\begin{aligned}
\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} &= \nabla_\sigma\nabla_\nu\nabla^\sigma\nabla_\nu C^{\mu\alpha\nu\beta} + D^{\alpha\beta} \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \nabla^\sigma[\nabla_\mu,\nabla_\sigma]\nabla_\nu\Box C^{\mu\alpha\nu\beta} + D^{\alpha\beta} \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} + D^{\alpha\beta} + \nabla^\sigma\Bigg[R_{\mu\sigma\nu}{}^{\lambda}\nabla_\lambda\Box C^{\mu\alpha\nu\beta} + R_{\mu\sigma}{}^{\mu}{}_{\lambda}\nabla_\nu\Box C^{\lambda\alpha\nu\beta} + R_{\mu\sigma}{}^{\alpha}{}_{\lambda}\nabla_\nu\Box \\
&\quad C^{\mu\lambda\nu\beta} + R_{\mu\sigma}{}^{\nu}{}_{\lambda}\nabla_\nu\Box C^{\mu\alpha\lambda\beta} + R_{\mu\sigma}{}^{\beta}{}_{\lambda}\nabla_\nu\Box C^{\mu\alpha\nu\lambda}\Bigg] \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} + D^{\alpha\beta} + \nabla^\sigma\Bigg[R_{\sigma\lambda}\nabla_\nu\Box C^{\lambda\alpha\nu\beta} - \frac{R}{12}\nabla_\mu\Box C^{\mu\alpha}{}_{\sigma}{}^{\beta} \\
&\quad + \frac{R}{12}(\nabla_\nu\Box C^{\alpha}{}_{\sigma}{}^{\nu\beta} + \nabla_\nu\Box C^{\nu\alpha}{}_{\sigma}{}^{\beta} + \nabla_\nu\Box C^{\beta\alpha\nu}{}_{\sigma})\Bigg] \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} + D^{\alpha\beta} + \nabla_\sigma\Bigg[-\frac{3}{l^2}\nabla_\nu\Box C^{\sigma\alpha\nu\beta} - \frac{R}{12}\nabla_\mu\Box C^{\mu\alpha\sigma\beta} + \frac{R}{12}\nabla_\nu\Box\overbrace{C^{\nu[\beta\alpha\sigma]}}^{=0}\Bigg] \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} - \frac{2}{l^2}\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} - \frac{3}{l^2}\nabla_\sigma\nabla_\nu\Box^{\sigma\alpha\nu\beta} + \frac{1}{l^2}\nabla_\sigma\nabla_\nu\Box C^{\mu\alpha\sigma\beta} \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} - \frac{5}{l^2}\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} + \frac{1}{l^2}\overbrace{\nabla_\sigma\nabla_\mu\Box C^{\sigma\beta\mu\alpha}}^{\nabla_\mu\nabla_\nu\Box C^{\mu\beta\nu\alpha}}\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} \\
&= \Box\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} - \frac{4}{l^2}\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} \\
&= (\Box - \frac{4}{l^2})\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta}
\end{aligned}
\tag{A.43}
$$

One can know that

$$
\nabla_\mu\nabla_\nu\Box C^{\mu\alpha\nu\beta} = \Big(\Box - \frac{4}{l^2}\Big)\nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} \tag{A.44}
$$

Therefore,

$$
\nabla_\mu\nabla_\nu\Box^2 C^{\mu\alpha\nu\beta} = \Big(\Box - \frac{4}{l^2}\Big)^2\nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} \tag{A.45}
$$

Hence, one can get the general form of these two last equations ,

$$
\nabla_\mu\nabla_\nu\Box^n C^{\mu\alpha\nu\beta} = \Big(\Box + \frac{R}{3}\Big)^n\nabla_\mu\nabla_\nu C^{\mu\alpha\nu\beta} = \frac{1}{2}\Big(\Box + \frac{R}{3}\Big)^n\Big(\Box - \frac{R}{3}\Big)S^{\alpha\beta} \tag{A.46}
$$